Negations in Description Logic – Contraries, Contradictories, and Subcontraries

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Abstract. In [10], several constructive description logics were proposed as intuitionistic variants of description logics, in which classical negation was replaced by strong negation as a component to treat negative atomic information. For conceptual representation, not only strong negation but also combining it with classical negation seems to be useful and necessary due to their respective features corresponding to predicate denial (e.g., not happy) and predicate term negation (e.g., unhappy). In this paper, we propose an alternative description logic $\mathcal{ALC}_\sim$ with classical negation and strong negation. In particular, we adhere to the notions of contraries, contradictories, and subcontraries (in [5]), generated from conceivable statement types using predicate denial and predicate term negation. To capture these notions, our formalization provides an improved semantics that suitably interprets various combinations of classical negation and strong negation. We show that our semantics preserves contradictoriness and contrariness for $\mathcal{ALC}_\sim$-concepts but the semantics of constructive description logic $\mathcal{CALC}_\sim$ with Heyting negation and strong negation cannot preserve the property for $\mathcal{CALC}_\sim$-concepts.

1 Introduction

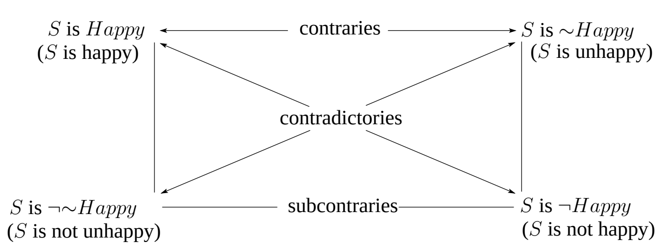
Negative information plays an important role in knowledge representation and reasoning (cf. [16, 8, 18]). Classical negation $\neg F$ represents the negation of a statement F, but a strong negation $\sim F$ may be more suitable for expressing explicit negative information (or negative facts). In other worlds, $\sim F$ indicates information that is directly opposite and exclusive to F rather than its complementary negation. Therefore the law of contradiction $\neg(F \wedge \sim F)$ holds but the law of excluded middle $F \vee \sim F$ does not hold [9, 1, 16]. For example, given the formula $rich(x)$ that represents "x is rich," the antonymous formula $poor(x)$ is defined by the strong negation $\sim rich(x)$, and not by the classical negation $\neg rich(x)$. Thus, we can logically recognize the inconsistency of $rich(x)$ and $\sim rich(x)$ (as $poor(x)$), and because $rich(x) \vee \sim rich(x)$ is not valid, we can represent "x is neither rich nor poor" as $\neg rich(x) \wedge \neg\sim rich(x)$.

In [10], several constructive description logics were proposed as intuitionistic variants of description logics, in which classical negation was replaced by strong negation as a component to treat negative atomic information. In contrast, since basic description logics correspond to a subclass of *classical* first-order logic (i.e., they have classical negation), negative concepts are expressed by classical negation. Due to the different features of negative information, complex negative statements using strong negation and classical negation can be usefully employed in conceptual representation. In the philosophical study of negation, there is a distinction between predicate denial (e.g., not happy) and predicate term negation (e.g., unhappy) [5, 18]. Moreover, conceivable statement types using predicate denial and predicate term negation give rise to opposition between affirmation and negation (contraries, contradictories, and subcontraries) [5]. When we logically establish classical negation and strong negation in concept languages, such philosophical notions are a rather suitable exposition of formalization.

In this paper, we propose a description logic $\mathcal{ALC}_\sim$ extended to include the two types of negation (classical negation $\neg$ and strong negation $\sim$), that is, an extension of the basic description logic $\mathcal{ALC}$. The following are the primary results of this paper. First, we present an improved semantics of $\mathcal{ALC}_\sim$, which adheres to oppositions – contraries, contradictories, and subcontraries in concept languages. We remark that the conventional semantics of strong negation [1] yields the undesirable equivalence $C \equiv {\sim}\neg C$ as opposed to our proposed semantics. Secondly, we show that our semantics preserves the property of contradictoriness and contrariness that there exists an element such that it belongs to the contradictory negation $\neg C$ but it does not belong to the contrary negation ${\sim}C$. When considering Heyting negation and classical negation in constructive description logics, the property cannot be preserved for some interpretations. The disadvantage motivates us to formalize a new semantics for contradictory negation and contrary negation. Based on the semantics, we develop a tableau-based algorithm for testing concept satisfiability in $\mathcal{ALC}_\sim$ and show the correctness (soundness, completeness, and termination) and complexity of the algorithm.

2 Contradictories and contraries between concepts

Strong negation can be used to describe conceptual oppositions in description logics such as the concepts $Happy$ and $Unhappy$. For example, let us denote by $Happy$, $\neg Happy$ (classical negation), and ${\sim}Happy$ (strong negation) the concepts "individuals that are happy," "individuals that are not happy," and "individuals that are unhappy," respectively. We can then construct the complex concepts $\exists has\text{-}child.\neg Happy$ as "Parents who have children that are not happy," $\exists has\text{-}child.{\sim}Happy$ as "Parents who have unhappy children," and $(\neg Happy \sqcap \neg{\sim}Happy) \sqcap Person$ as "Persons who are neither happy nor unhappy." Syntactically, these allow us to express concepts composed of various combinations of classical negation and strong negation, e.g., ${\sim}\neg C$ and ${\sim}\neg{\sim}C$. As discussed in [12], the two negations represent the following oppositions between affirmation and negation (which Horn [5] renders):



In our conceptual explanation of them, the contraries ($Happy$ and $\sim Happy$) imply that both concepts cannot contain an identical element. The contradictories ($Happy$ and $\neg Happy$) imply that one concept must contain an element when it does not belong to the other. The subcontraries ($\neg\sim Happy$ and $\neg Happy$) imply that every element belongs to either of the concepts. In order to apply the oppositions to any DL-concepts, we require to generalize them as follows [1]:

$(\neg\sim)^i A, \sim(\neg\sim)^i A$ and $\neg(\sim\neg)^i A, (\sim\neg)^{i+1} A$:	contraries
$(\sim\neg)^i A, \neg(\sim\neg)^i A$ and $\sim(\neg\sim)^i A, (\neg\sim)^{i+1} A$:	contradictories
$(\neg\sim)^{i+1} A, \neg(\neg\sim)^i A$ and $\neg(\sim\neg)^{i+1} A, (\sim\neg)^i A$:	subcontraries

where A is a concept name (i.e., an atomic concept). In reasoning algorithms for description logics with the two negations, contraries and contradictories will be taken as a criterion of checking inconsistent pairs of DL-concepts.

In intuitionistic logic, Heyting negation and strong negation exist as methods of dealing with the oppositions. In general, strong negation has been formalized as a constructive negation of intuitionistic logic. Thomason [14] proposed the semantics of intuitionistic first-order logic only with strong negation, Akama [1] formalized intuitionistic logic with Heyting negation and strong negation, Wagner [15] and Herre et al.[3] defined weak negation and strong negation in the logic developed by them, and Pearce and Wagner [11] proposed logic programming with strong negation that was regarded as a subsystem of intuitionistic logic. While intuitionistic logic and strong negation allow us to represent term negation and predicate denial, we would like to propose a description logic such that:

1. It contains classical negation and strong negation since $\mathcal{ALC}$ is based on classical logic.
2. It fulfills the property that contradictoriness and contrariness are preserved for every interpretation.

Here, we observe the properties of Heyting negation ($-$) and strong negation ($\sim$). Let F be a formula (but not a concept). The law of double negation $--F \leftrightarrow F$ and the law of excluded middle $-F \vee F$ do not hold for Heyting negation, but the law of double negation $\sim\sim F \leftrightarrow F$ is valid for strong negation. The combinations of these negations lead to the valid axiom $\sim - F \leftrightarrow F$. Hence, the contradictory F and $-F$ can be replaced with $\sim - F$ and $-F$ by the equivalence $\sim - F \leftrightarrow F$; however, it should be recognized as a contrary. This is in partial disagreement with the abovementioned oppositions in which contradictories and contraries are defined differently. With regard to these features, Heyting negation and strong negation using the conventional semantics in intuitionistic logic [14, 1, 3] do not satisfy our requirements. Thus, we need to model contradictory negation and contrary negation in description logics and compare it with the constructive description logics in [10].

To incorporate strong negation into classical first-order logic and to remove the equivalence $\sim\neg F \leftrightarrow F$, we have improved the semantics by capturing the following

[1] $(\sim\neg)^i$ (resp. $(\neg\sim)^i$) denotes a chain of length i of $\sim\neg$ (resp. $\neg\sim$).

properties [7]. The law of double negation $\neg\neg F$ and the law of excluded middle $\neg F \vee F$ hold for classical negation, and the equivalence $\sim\neg F \leftrightarrow F$ is not valid. In conceptual representation based on this semantics, the strong negation $\sim C$ of a concept C is partial and exclusive to its affirmative expression C. The partiality of strong negation entails the existence of information that is neither affirmative nor strongly negative, i.e., $\neg C \sqcap \neg\sim C \not\equiv \bot$. In contrast, the classical negation $\neg C$ is complementary and exclusive to its affirmative expression C. Hence, the disjunction of affirmation and its classical negation expresses the set of all individuals, i.e., $C \sqcup \neg C \equiv \top$. Additionally, the simple double negations $\neg\neg C$ and $\sim\sim C$ are interpreted to be equivalent to the affirmation C. We can constructively define the complicated double negations $\sim\neg C$ and $\neg\sim C$ without losing the features of the two negations by the refinement of the conventional semantics. If we strongly deny the classical negation $\neg C$, then the double negation $\sim\neg C$ (called *constructive double negation*) must be partial and exclusive to $\neg C$. If we express the classical negation of a strong negation $\sim C$, then the double negation $\neg\sim C$ (called *weak double negation*) must be complementary and exclusive to $\sim C$.

3 Strong negation in description logic

In this section, we define a description logic with classical negation and strong negation that is interpreted by two different semantics and analyze the property of contradictoriness and contrariness for the proposed logic. In addition, we define a constructive description logic obtained by including Heyting negation and strong negation.

3.1 The description logic with classical negation and strong negation: $\mathcal{ALC}_\sim$

The description logic $\mathcal{ALC}_\sim$ (as an extension of $\mathcal{ALC}$ [13]) is based upon a set $\mathbf{C}$ of concept names A (including $\top$ and $\bot$), a set $\mathbf{R}$ of role names R, and a set $\mathbf{I}$ of individual names a. The concepts of the language (called $\mathcal{ALC}_\sim$-concepts) are constructed by concept names A; role names R; the connectives $\sqcap$, $\sqcup$, $\neg$ (classical negation), and $\sim$ (strong negation); and the quantifiers $\forall$ and $\exists$. Every concept name $A \in \mathbf{C}$ is an $\mathcal{ALC}_\sim$-concept. If R is a role name and C, D are $\mathcal{ALC}_\sim$-concepts, then $\neg C$, $\sim C$, $C \sqcap D$, $C \sqcup D$, $\forall R.C$, and $\exists R.C$ are $\mathcal{ALC}_\sim$-concepts.

We denote as $sub(C)$ the set of subconcepts of an $\mathcal{ALC}_\sim$-concept C. Let X be a sequence of classical negation $\neg$ and strong negation $\sim$. We denote $(X)^n$ as a chain of length n of X. For instance, $\sim(\neg\sim)^2 C_1$ and $(\sim\neg)^0 C_2$ denote $\sim\neg\sim\neg\sim C_1$ and C_2, respectively. Next, we define an interpretation of $\mathcal{ALC}_\sim$-concepts (called an $\mathcal{ALC}^2_\sim$-interpretation) by using the conventional semantics of strong negation.

Definition 1. *An $\mathcal{ALC}^2_\sim$-interpretation $\mathcal{I}$ is a tuple $(\Delta^\mathcal{I}, \cdot^{\mathcal{I}^+}, \cdot^{\mathcal{I}^-})$, where $\Delta^\mathcal{I}$ is a non-empty set and $\cdot^{\mathcal{I}^+}$ and $\cdot^{\mathcal{I}^-}$ are interpretation functions ($A^{\mathcal{I}^+} \subseteq \Delta^\mathcal{I}$, $A^{\mathcal{I}^-} \subseteq \Delta^\mathcal{I}$, $R^{\mathcal{I}^+} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$, and $a^{\mathcal{I}^+} \in \Delta^\mathcal{I}$) such that:*

1. $\bot^{\mathcal{I}^+} = \emptyset$ *and* $\top^{\mathcal{I}^+} = \Delta^\mathcal{I}$,
2. $A^{\mathcal{I}^+} \cap A^{\mathcal{I}^-} = \emptyset$.

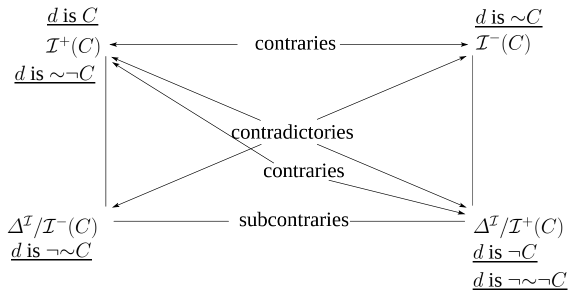


Fig. 1. Oppositions in $\mathcal{ALC}^2_\sim$-interpretations

The interpretation functions $\cdot^{\mathcal{I}^+}$ *and* $\cdot^{\mathcal{I}^-}$ *are expanded to* $\mathcal{ALC}_\sim$*-concepts as follows:*

$$
\begin{aligned}
(\neg C)^{\mathcal{I}^+} &= \Delta^{\mathcal{I}} \backslash C^{\mathcal{I}^+} & (\sim C)^{\mathcal{I}^+} &= C^{\mathcal{I}^-} \\
(C \sqcap D)^{\mathcal{I}^+} &= C^{\mathcal{I}^+} \cap D^{\mathcal{I}^+} & (C \sqcup D)^{\mathcal{I}^+} &= C^{\mathcal{I}^+} \cup D^{\mathcal{I}^+} \\
(\forall R.C)^{\mathcal{I}^+} &= \{d_1 \in \Delta^{\mathcal{I}} \mid \forall d_2[(d_1, d_2) \in R^{\mathcal{I}^+} \to d_2 \in C^{\mathcal{I}^+}]\} \\
(\exists R.C)^{\mathcal{I}^+} &= \{d_1 \in \Delta^{\mathcal{I}} \mid \exists d_2[(d_1, d_2) \in R^{\mathcal{I}^+} \wedge d_2 \in C^{\mathcal{I}^+}]\} \\
(\neg C)^{\mathcal{I}^-} &= C^{\mathcal{I}^+} & (\sim C)^{\mathcal{I}^-} &= C^{\mathcal{I}^+} \\
(C \sqcap D)^{\mathcal{I}^-} &= C^{\mathcal{I}^-} \cup D^{\mathcal{I}^-} & (C \sqcup D)^{\mathcal{I}^-} &= C^{\mathcal{I}^-} \cap D^{\mathcal{I}^-} \\
(\forall R.C)^{\mathcal{I}^-} &= \{d_1 \in \Delta^{\mathcal{I}} \mid \exists d_2[(d_1, d_2) \in R^{\mathcal{I}^+} \wedge d_2 \in C^{\mathcal{I}^-}]\} \\
(\exists R.C)^{\mathcal{I}^-} &= \{d_1 \in \Delta^{\mathcal{I}} \mid \forall d_2[(d_1, d_2) \in R^{\mathcal{I}^+} \to d_2 \in C^{\mathcal{I}^-}]\}
\end{aligned}
$$

An $\mathcal{ALC}^2_\sim$-interpretation $\mathcal{I}$ satisfies the contrary condition if for all concept names A, $A^{\mathcal{I}^+} \cup A^{\mathcal{I}^-} \neq \Delta^{\mathcal{I}}$. The $\mathcal{ALC}^2_\sim$-interpretation is defined by the two interpretation functions $\cdot^{\mathcal{I}^+}$ and $\cdot^{\mathcal{I}^-}$, but it causes an undesirable equation $C \equiv \sim\neg C$, i.e., $C^{\mathcal{I}^+} = (\sim\neg C)^{\mathcal{I}^+}$. Contradictoriness and contrariness are preserved in an $\mathcal{ALC}^2_\sim$-interpretation $\mathcal{I}$ of $\mathcal{ALC}_\sim$ if $\sim C^{\mathcal{I}} \subsetneq \neg C^{\mathcal{I}}$. The interpretation results in the following negative property:

Theorem 1 (Contradictoriness and contrariness for $\mathcal{ALC}^2_\sim$).
Contradictoriness and contrariness are not preserved in every $\mathcal{ALC}^2_\sim$-interpretation that satisfies the contrary condition.

Subsequently, we define an alternative interpretation of $\mathcal{ALC}_\sim$-concepts (called an $\mathcal{ALC}^n_\sim$-interpretation), which is based on the semantics [7] obtained by improving Akama's semantics [1].

Definition 2. *An $\mathcal{ALC}^n_\sim$-interpretation $\mathcal{I}$ is a tuple $(\Delta^{\mathcal{I}}, \{\cdot^{\mathcal{I}_i^+} \mid i \in \omega\}, \{\cdot^{\mathcal{I}_i^-} \mid i \in \omega\})$,* [2] *where $\Delta^{\mathcal{I}}$ is a non-empty set and $\cdot^{\mathcal{I}_i^+}$ and $\cdot^{\mathcal{I}_i^-}$ are interpretation functions ($A^{\mathcal{I}_i^+} \subseteq \Delta^{\mathcal{I}}$, $A^{\mathcal{I}_i^-} \subseteq \Delta^{\mathcal{I}}$, $R^{\mathcal{I}_0^+} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, and $a^{\mathcal{I}_0^+} \in \Delta^{\mathcal{I}}$), such that:*

[2] The symbol ω denotes the set of natural numbers. Thus, $\{\cdot^{\mathcal{I}_i^+} \mid i \in \omega\}$ is infinite as $\{\cdot^{\mathcal{I}_0^+}, \cdot^{\mathcal{I}_1^+}, \cdot^{\mathcal{I}_2^+}, \cdot^{\mathcal{I}_3^+}, \dots\}$.

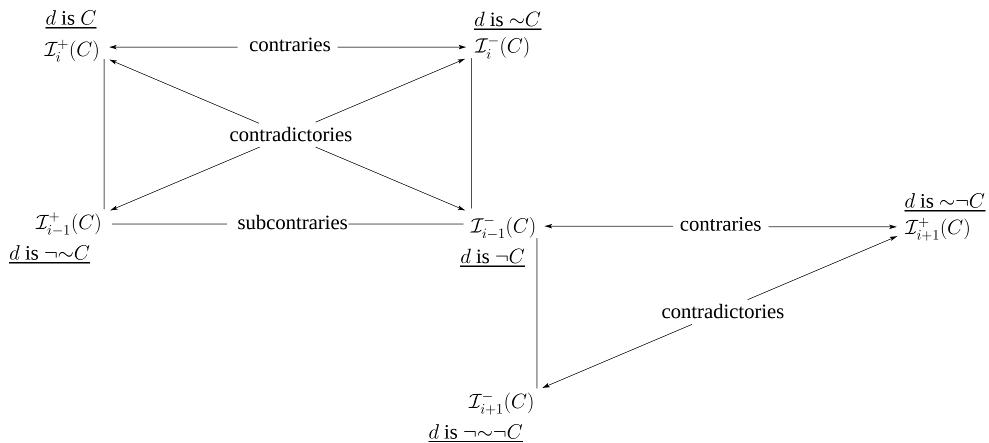


Fig. 2. Oppositions in $\mathcal{ALC}_\sim^n$-interpretations

1. $\bot^{\mathcal{I}_0^+} = \emptyset$ *and* $\top^{\mathcal{I}_0^+} = \Delta^{\mathcal{I}}$*,*
2. $A^{\mathcal{I}_0^+} \cap A^{\mathcal{I}_0^-} = \emptyset$*,*
3. $A^{\mathcal{I}_{i+1}^+} \subseteq A^{\mathcal{I}_i^+}$ *and* $A^{\mathcal{I}_{i+1}^-} \subseteq A^{\mathcal{I}_i^-}$*.*

The interpretation functions $\cdot^{\mathcal{I}_i^+}$ *and* $\cdot^{\mathcal{I}_i^-}$ *are expanded to* $\mathcal{ALC}_\sim$*-concepts as follows:*

$$
\begin{array}{rlrl}
(\neg C)^{\mathcal{I}_0^+} &= \Delta^{\mathcal{I}} \backslash C^{\mathcal{I}_0^+} & & \\
(\neg C)^{\mathcal{I}_i^+} &= C^{\mathcal{I}_{i-1}^-} (i > 0) & (\sim C)^{\mathcal{I}_i^+} &= C^{\mathcal{I}_i^-} \\
(C \sqcap D)^{\mathcal{I}_i^+} &= C^{\mathcal{I}_i^+} \cap D^{\mathcal{I}_i^+} & (C \sqcup D)^{\mathcal{I}_i^+} &= C^{\mathcal{I}_i^+} \cup D^{\mathcal{I}_i^+} \\
(\forall R.C)^{\mathcal{I}_i^+} &= \{d_1 \in \Delta^{\mathcal{I}} \mid \forall d_2[(d_1, d_2) \in R^{\mathcal{I}_0^+} \to d_2 \in C^{\mathcal{I}_i^+}]\} & & \\
(\exists R.C)^{\mathcal{I}_i^+} &= \{d_1 \in \Delta^{\mathcal{I}} \mid \exists d_2[(d_1, d_2) \in R^{\mathcal{I}_0^+} \land d_2 \in C^{\mathcal{I}_i^+}]\} & & \\
(\neg C)^{\mathcal{I}_i^-} &= C^{\mathcal{I}_{i+1}^+} & (\sim C)^{\mathcal{I}_i^-} &= C^{\mathcal{I}_i^+} \\
(C \sqcap D)^{\mathcal{I}_i^-} &= C^{\mathcal{I}_i^-} \cup D^{\mathcal{I}_i^-} & (C \sqcup D)^{\mathcal{I}_i^-} &= C^{\mathcal{I}_i^-} \cap D^{\mathcal{I}_i^-} \\
(\forall R.C)^{\mathcal{I}_i^-} &= \{d_1 \in \Delta^{\mathcal{I}} \mid \exists d_2[(d_1, d_2) \in R^{\mathcal{I}_0^+} \land d_2 \in C^{\mathcal{I}_i^-}]\} & & \\
(\exists R.C)^{\mathcal{I}_i^-} &= \{d_1 \in \Delta^{\mathcal{I}} \mid \forall d_2[(d_1, d_2) \in R^{\mathcal{I}_0^+} \to d_2 \in C^{\mathcal{I}_i^-}]\} & &
\end{array}
$$

An $\mathcal{ALC}_\sim^n$-interpretation $\mathcal{I}$ satisfies the contrary condition if for all concept names A, $A^{\mathcal{I}_0^+} \cup A^{\mathcal{I}_0^-} \neq \Delta^{\mathcal{I}}$, $A^{\mathcal{I}_{i+1}^+} \subsetneq A^{\mathcal{I}_i^+}$, and $A^{\mathcal{I}_{i+1}^-} \subsetneq A^{\mathcal{I}_i^-}$. In the two types of interpretations, conceptual oppositions are characterized as shown in Fig.1 and Fig.2. The $\mathcal{ALC}_\sim^2$-interpretation is defined as $(\sim\neg C)^{\mathcal{I}^+} = (\neg C)^{\mathcal{I}^-} = C^{\mathcal{I}^+}$, and hence, $d \in (\sim\neg C)^{\mathcal{I}}$ if and only if $d \in C^{\mathcal{I}}$. This semantically causes loss in distinction between contraries ($\sim\neg C$ and $\neg C$) and contradictories (C and $\neg C$). Instead, the $\mathcal{ALC}_\sim^n$-interpretation includes the definition $(\sim\neg C)^{\mathcal{I}_i^+} = (\neg C)^{\mathcal{I}_i^-} = C^{\mathcal{I}_{i+1}^+}$ and $(\neg C)^{\mathcal{I}_i^+} = C^{\mathcal{I}_{i-1}^-}$ ($i > 0$), where infinite interpretation functions are required. That is, the $\mathcal{ALC}_\sim^n$-interpretation is improved in order to capture the oppositions – contraries, contradictories, and subcontraries in the philosophical study of negation [5].

Each $\mathcal{ALC}^n_\sim$-interpretation (resp. $\mathcal{ALC}^2_\sim$-interpretation) $C^\mathcal{I}$ of each $\mathcal{ALC}_\sim$-concept is given by $C^{\mathcal{I}_0^+}$ (resp. $C^{\mathcal{I}^+}$). An $\mathcal{ALC}_\sim$-concept C (or a concept equation $C \equiv D$) is $\mathcal{ALC}^n_\sim$-satisfiable (resp. $\mathcal{ALC}^2_\sim$-satisfiable) if there exists an $\mathcal{ALC}^n_\sim$-interpretation (resp. $\mathcal{ALC}^2_\sim$-interpretation) $\mathcal{I}$, called an $\mathcal{ALC}^n_\sim$-model (resp. $\mathcal{ALC}^2_\sim$-model) of C (or $C \equiv D$), such that $C^\mathcal{I} \neq \emptyset$ (or $C^\mathcal{I} = D^\mathcal{I}$). Otherwise, it is $\mathcal{ALC}^n_\sim$-unsatisfiable (resp. $\mathcal{ALC}^2_\sim$-unsatisfiable). In particular, if an $\mathcal{ALC}_\sim$-concept C is $\mathcal{ALC}^n_\sim$-satisfiable (resp. $\mathcal{ALC}^2_\sim$-satisfiable) and the $\mathcal{ALC}^n_\sim$-model (resp. $\mathcal{ALC}^2_\sim$-model) satisfies the contrary condition, then it is $\mathcal{ALC}^n_\sim$-satisfiable (resp. $\mathcal{ALC}^2_\sim$-satisfiable) under the contrary condition. Otherwise, it is $\mathcal{ALC}^n_\sim$-unsatisfiable (resp. $\mathcal{ALC}^2_\sim$-unsatisfiable) under the contrary condition. A concept equation $C \equiv D$ is $\mathcal{ALC}^n_\sim$-valid (resp. $\mathcal{ALC}^2_\sim$-valid) if every $\mathcal{ALC}^n_\sim$-interpretation ($\mathcal{ALC}^2_\sim$-interpretation) $\mathcal{I}$ is an $\mathcal{ALC}^n_\sim$-model ($\mathcal{ALC}^2_\sim$-model) of $C \equiv D$. We can derive the following fact from these interpretations:

Proposition 1. *If C is an $\mathcal{ALC}_\sim$-concept, then the concept equation $C \equiv \sim\neg C$ is not $\mathcal{ALC}^n_\sim$-valid, but it is $\mathcal{ALC}^2_\sim$-valid.*

In addition, since $\mathcal{ALC}_\sim$-concepts do not contain the negation of roles, each role is interpreted only by the interpretation function $\cdot^{\mathcal{I}_0^+}$ (or $\cdot^{\mathcal{I}^+}$). Let us give an example of an $\mathcal{ALC}^n_\sim$-interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \{\cdot^{\mathcal{I}_i^+} \mid i \in \omega\}, \{\cdot^{\mathcal{I}_i^-} \mid i \in \omega\})$ such that $\Delta^\mathcal{I} = \{John, Mary, Tom\}$, $Happy^{\mathcal{I}_0^+} = \{John\}$, $Happy^{\mathcal{I}_0^-} = \{Mary, Tom\}$, $Happy^{\mathcal{I}_1^+} = \emptyset$, $Happy^{\mathcal{I}_1^-} = \{Tom\}$, $Happy^{\mathcal{I}_2^+} = \emptyset$, $\ldots$, $has\text{-}child^{\mathcal{I}_0^+} = \{(John, Tom)\}$ with $Happy^{\mathcal{I}_0^+} \cap Happy^{\mathcal{I}_0^-} = \emptyset$, $Happy^{\mathcal{I}_{i+1}^+} \subseteq Happy^{\mathcal{I}_i^+}$, and $Happy^{\mathcal{I}_{i+1}^-} \subseteq Happy^{\mathcal{I}_i^-}$. Then, we obtain the interpretation functions $\cdot^{\mathcal{I}_i^+}$ and $\cdot^{\mathcal{I}_i^-}$ expanded to the $\mathcal{ALC}_\sim$-concepts $\exists has\text{-}child.{\sim}Happy$, $\neg\sim\neg\sim Happy$, and $\neg\sim\sim\neg Happy$ as below:

$$\begin{aligned}(\exists has\text{-}child.{\sim}Happy)^{\mathcal{I}_0^+} &= \{d_1 \in \Delta^\mathcal{I} \mid \exists d_2[(d_1, d_2) \in has\text{-}child^{\mathcal{I}_0^+} \wedge d_2 \in Happy^{\mathcal{I}_0^-}]\} \\ &= \{John\}\end{aligned}$$

$$\begin{aligned}(\neg\sim\neg\sim Happy)^{\mathcal{I}_0^+} &= \Delta^\mathcal{I} \backslash (\sim\neg\sim Happy)^{\mathcal{I}_0^+} & (\neg\sim\sim\neg Happy)^{\mathcal{I}_0^+} &= \Delta^\mathcal{I} \backslash (\sim\sim\neg Happy)^{\mathcal{I}_0^+} \\ &= \Delta^\mathcal{I} \backslash (\neg\sim Happy)^{\mathcal{I}_0^-} & &= \Delta^\mathcal{I} \backslash (\sim\neg Happy)^{\mathcal{I}_0^-} \\ &= \Delta^\mathcal{I} \backslash (\sim Happy)^{\mathcal{I}_1^+} & &= \Delta^\mathcal{I} \backslash (\neg Happy)^{\mathcal{I}_0^+} \\ &= \Delta^\mathcal{I} \backslash Happy^{\mathcal{I}_1^-} & &= \{John\} \\ &= \{John, Mary\} & & \end{aligned}$$

Remark. Semantically, the three conditions $A^{\mathcal{I}_0^+} \cap A^{\mathcal{I}_0^-} = \emptyset$, $A^{\mathcal{I}_{i+1}^+} \subseteq A^{\mathcal{I}_i^+}$, and $A^{\mathcal{I}_{i+1}^-} \subseteq A^{\mathcal{I}_i^-}$ in the $\mathcal{ALC}^n_\sim$-interpretation $\mathcal{I}$ define the inconsistency of contraries between $\mathcal{ALC}_\sim$-concepts. Syntactically, the conditions lead to the inconsistent pairs $\langle A, \sim A\rangle$, $\langle \neg(\sim\neg)^i A, (\sim\neg)^{i+1} A\rangle$, and $\langle (\neg\sim)^{i+1} A, \sim(\neg\sim)^{i+1} A\rangle$ of $\mathcal{ALC}_\sim$-concepts. Each pair consists of a concept C and its strong negation $\sim C$ (i.e., $\langle C, \sim C\rangle$) where C is of the form A, $\neg(\sim\neg)^i A$, or $(\neg\sim)^{i+1} A$. For example, $\neg Red$ and $\sim\neg Red$ are inconsistent. In the next lemma, these conditions are generalized to any $\mathcal{ALC}_\sim$-concept.

Lemma 1. *Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \{\cdot^{\mathcal{I}_i^+} \mid i \in \omega\}, \{\cdot^{\mathcal{I}_i^-} \mid i \in \omega\})$ be an $\mathcal{ALC}^n_\sim$-interpretation. For any $\mathcal{ALC}_\sim$-concept C, the following statements hold:*

1. $C^{\mathcal{I}} \cap {\sim}C^{\mathcal{I}} = \emptyset$,
2. $({\sim}\neg)^{i+1}C^{\mathcal{I}} \subseteq ({\sim}\neg)^{i}C^{\mathcal{I}}$,
3. ${\sim}(\neg{\sim})^{i+1}C^{\mathcal{I}} \subseteq {\sim}(\neg{\sim})^{i}C^{\mathcal{I}}$.

This lemma will be used to prove the correspondence between a tableau for an $\mathcal{ALC}_\sim$-concept and the satisfiability of the concept.

Any $\mathcal{ALC}_\sim$-concept is transformed into an equivalent one in a normal negation form (that is more complicated than the normal negation form in $\mathcal{ALC}$) using the following equivalences from left to right:

$$
\begin{aligned}
(\neg)^k({\sim}\neg)^i{\sim}{\sim}C &\equiv (\neg)^k({\sim}\neg)^iC\\
({\sim})^k(\neg{\sim})^i\neg\neg C &\equiv ({\sim})^k(\neg{\sim})^iC\\
(\neg)^k({\sim}\neg)^i{\sim}(C \sqcap D) &\equiv (\neg)^k({\sim}\neg)^i({\sim}C \sqcup {\sim}D)\\
(\neg)^k({\sim}\neg)^i{\sim}(C \sqcup D) &\equiv (\neg)^k({\sim}\neg)^i({\sim}C \sqcap {\sim}D)\\
(\neg)^k({\sim}\neg)^i{\sim}(\forall R.C) &\equiv (\neg)^k({\sim}\neg)^i(\exists R.{\sim}C)\\
(\neg)^k({\sim}\neg)^i{\sim}(\exists R.C) &\equiv (\neg)^k({\sim}\neg)^i(\forall R.{\sim}C)\\
({\sim})^k(\neg{\sim})^i\neg(C \sqcap D) &\equiv ({\sim})^k(\neg{\sim})^i(\neg C \sqcup \neg D)\\
({\sim})^k(\neg{\sim})^i\neg(C \sqcup D) &\equiv ({\sim})^k(\neg{\sim})^i(\neg C \sqcap \neg D)\\
({\sim})^k(\neg{\sim})^i\neg(\forall R.C) &\equiv ({\sim})^k(\neg{\sim})^i(\exists R.\neg C)\\
({\sim})^k(\neg{\sim})^i\neg(\exists R.C) &\equiv ({\sim})^k(\neg{\sim})^i(\forall R.\neg C)
\end{aligned}
$$

where $k \in \{0,1\}$ and $i \in \omega$. The form of the concepts obtained by this transformation is called a *constructive normal negation form*, where the four types of negation forms $({\sim}\neg)^{i+1}$, $\neg({\sim}\neg)^i$, $(\neg{\sim})^{i+1}$, and ${\sim}(\neg{\sim})^i$ occur only in front of a concept name. For example, $({\sim}\neg A_1 \sqcup \neg{\sim}\neg A_2) \sqcap {\sim}(\neg{\sim})^4 A_3$ is in the constructive normal negation form.

Proposition 2. *Every concept equation $C \equiv D$ in the translation is $\mathcal{ALC}^n_\sim$-valid.*

Next, we will discuss an important property of $\mathcal{ALC}^n_\sim$-interpretations that is derived from the contrary condition.

Lemma 2. *Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \{\cdot^{\mathcal{I}_i^+} \mid i \in \omega\}, \{\cdot^{\mathcal{I}_i^-} \mid i \in \omega\})$ be an $\mathcal{ALC}^n_\sim$-interpretation that satisfies the contrary condition. For any $\mathcal{ALC}_\sim$-concept C, the following statements hold:*

1. $C^{\mathcal{I}} \cup {\sim}C^{\mathcal{I}} \neq \Delta^{\mathcal{I}}$,
2. $({\sim}\neg)^{i+1}C^{\mathcal{I}} \subsetneq ({\sim}\neg)^{i}C^{\mathcal{I}}$,
3. ${\sim}(\neg{\sim})^{i+1}C^{\mathcal{I}} \subsetneq {\sim}(\neg{\sim})^{i}C^{\mathcal{I}}$.

This lemma guarantees that the proposed semantics characterizes the differences between contradictories and contraries in every interpretation. The following theorem states the property of contradictoriness and contrariness for $\mathcal{ALC}^n_\sim$-interpretations. Contradictoriness and contrariness are preserved in an $\mathcal{ALC}^n_\sim$-interpretation $\mathcal{I}$ of $\mathcal{ALC}_\sim$ if ${\sim}C^{\mathcal{I}} \subsetneq \neg C^{\mathcal{I}}$.

Theorem 2 (Contradictoriness and contrariness for $\mathcal{ALC}^n_\sim$).
If an $\mathcal{ALC}^n_\sim$-interpretation satisfies the contrary condition, then contradictoriness and contrariness are preserved in the $\mathcal{ALC}^n_\sim$-interpretation.

We would like to apply the replacement property [17] to conceptual representation and strong negation in $\mathcal{ALC}_\sim$. Knowledge base designers rewrite concepts by their equivalent concepts in the context of (conceptual) knowledge representation (e.g., rebuilding ontologies in the Semantic Web). However, the replacement property provides a limitation such that concepts can only be replaced by strongly equivalent concepts when various combinations of the two types of negation are used. Let $C,\ D$ be $\mathcal{ALC}_\sim$-concepts. C and D are equivalent if, for every $\mathcal{ALC}^n_\sim$-interpretation $\mathcal{I}$, $C^\mathcal{I} = D^\mathcal{I}$. C and D are strongly equivalent if, for every $\mathcal{ALC}^n_\sim$-interpretation $\mathcal{I}$, $C^{\mathcal{I}_i^+} = D^{\mathcal{I}_i^+}$ and $C^{\mathcal{I}_i^-} = D^{\mathcal{I}_i^-}$. The concept E_C denotes a (complex) concept that contains the concept C as a subconcept of E_C and E_D denotes the concept obtained by replacing C in E_C by D.

Theorem 3 (Replacement for $\mathcal{ALC}_\sim$). *Let $C,\ D$ be $\mathcal{ALC}_\sim$-concepts. If C and D are strongly equivalent, then E_C and E_D are also equivalent.*

It should be noted that the replacement property under strong equivalence is natural in the presence of strong negation.

3.2 The constructive description logic with Heyting negation and strong negation: $\mathcal{CALC}_\sim$

We define the description logic $\mathcal{CALC}_\sim$ (as an extension of the constructive description logic $\mathcal{CALC}^{N4}$ [10]) by combining Heyting negation and strong negation. The concepts in the language (called $\mathcal{CALC}_\sim$-concepts) are constructed by concept names A; role names R; the connectives $\sqcap$, $\sqcup$, $-$ (Heyting negation), and $\sim$ (strong negation); and the quantifiers $\forall$, $\exists$. Every concept name $A \in \mathbf{C}$ is a $\mathcal{CALC}_\sim$-concept. If R is a role name and C, D are $\mathcal{CALC}_\sim$-concepts, then $-C$, $\sim C$, $C \sqcap D$, $C \sqcup D$, $\forall R.C$, and $\exists R.C$ are $\mathcal{CALC}_\sim$-concepts. We give an interpretation of $\mathcal{CALC}_\sim$-concepts (called a $\mathcal{CALC}^2_\sim$-interpretation) as follows:

Definition 3. *A $\mathcal{CALC}^2_\sim$-interpretation $\mathcal{I}$ is a tuple $(W, \preceq, \Delta^\mathcal{I}, \{\cdot^{\mathcal{I}_t^+} \mid t \in W\}, \{\cdot^{\mathcal{I}_t^-} \mid t \in W\})$, where W is a set of worlds, $\Delta^\mathcal{I} = \{\Delta^{\mathcal{I}_t} \mid t \in W\}$ is the family of non-empty sets and $\cdot^{\mathcal{I}_t^+}$ and $\cdot^{\mathcal{I}_t^-}$ are interpretation functions for each world $t \in W$ ($A^{\mathcal{I}_t^+} \subseteq \Delta^\mathcal{I}$, $A^{\mathcal{I}_t^-} \subseteq \Delta^\mathcal{I}$, $R^{\mathcal{I}_t^+} \subseteq \Delta^{\mathcal{I}_t} \times \Delta^{\mathcal{I}_t}$, and $a^{\mathcal{I}_t^+} \in \Delta^{\mathcal{I}_t}$) such that:*

1. $\bot^{\mathcal{I}_t^+} = \emptyset$ *and* $\top^{\mathcal{I}_t^+} = \Delta^{\mathcal{I}_t}$,
2. $A^{\mathcal{I}_t^+} \cap A^{\mathcal{I}_t^-} = \emptyset$,
3. *if* $t, t' \in W$ *and* $t \preceq t'$, *then* $\Delta^{\mathcal{I}_t} \subseteq \Delta^{\mathcal{I}_{t'}}$, $A^{\mathcal{I}_t^+} \subseteq A^{\mathcal{I}_{t'}^+}$, $A^{\mathcal{I}_t^-} \subseteq A^{\mathcal{I}_{t'}^-}$, *and* $R^{\mathcal{I}_t^+} \subseteq R^{\mathcal{I}_{t'}^+}$.

For every world $t \in W$, the interpretation functions $\cdot^{\mathcal{I}_t^+}$ and $\cdot^{\mathcal{I}_t^-}$ are expanded to $\mathcal{CALC}_\sim$-concepts as follows:

$$
\begin{aligned}
(-C)^{\mathcal{I}_t^+} &= \{d \,|\, d \in \Delta^{\mathcal{I}_{t'}} \backslash C^{\mathcal{I}_{t'}^+} \text{ s.t. } t \preceq t'\} & (\sim C)^{\mathcal{I}_t^+} &= C^{\mathcal{I}_t^-} \\
(C \sqcap D)^{\mathcal{I}_t^+} &= C^{\mathcal{I}_t^+} \cap D^{\mathcal{I}_t^+} & (C \sqcup D)^{\mathcal{I}_t^+} &= C^{\mathcal{I}_t^+} \cup D^{\mathcal{I}_t^+} \\
(\forall R.C)^{\mathcal{I}_t^+} &= \{d_1 \in \Delta^{\mathcal{I}_t} \,|\, \forall t'[t \preceq t' \to \forall d_2 \in \Delta^{\mathcal{I}_{t'}}[(d_1, d_2) \in R^{\mathcal{I}_{t'}^+} \to d_2 \in C^{\mathcal{I}_{t'}^+}]]\} \\
(\exists R.C)^{\mathcal{I}_t^+} &= \{d_1 \in \Delta^{\mathcal{I}_t} \,|\, \exists d_2 \in \Delta^{\mathcal{I}_{t'}}[(d_1, d_2) \in R^{\mathcal{I}_t^+} \wedge d_2 \in C^{\mathcal{I}_t^+}]\} \\
(-C)^{\mathcal{I}_t^-} &= C^{\mathcal{I}_t^+} & (\sim C)^{\mathcal{I}_t^-} &= C^{\mathcal{I}_t^+} \\
(C \sqcap D)^{\mathcal{I}_t^-} &= C^{\mathcal{I}_t^-} \cup D^{\mathcal{I}_t^-} & (C \sqcup D)^{\mathcal{I}_t^-} &= C^{\mathcal{I}_t^-} \cap D^{\mathcal{I}_t^-} \\
(\forall R.C)^{\mathcal{I}_t^-} &= \{d_1 \in \Delta^{\mathcal{I}_t} \,|\, \exists d_2 \in \Delta^{\mathcal{I}_{t'}}[(d_1, d_2) \in R^{\mathcal{I}_t^+} \wedge d_2 \in C^{\mathcal{I}_t^-}]\} \\
(\exists R.C)^{\mathcal{I}_t^-} &= \{d_1 \in \Delta^{\mathcal{I}_t} \,|\, \forall t'[t \preceq t' \to \forall d_2 \in \Delta^{\mathcal{I}_{t'}}[(d_1, d_2) \in R^{\mathcal{I}_{t'}^+} \to d_2 \in C^{\mathcal{I}_{t'}^-}]]\}
\end{aligned}
$$

An $\mathcal{ALC}^n_\sim$-interpretation $\mathcal{I}$ satisfies the contrary condition if $A^{\mathcal{I}^+} \cup A^{\mathcal{I}^-} \neq \Delta^{\mathcal{I}}$, where $A^{\mathcal{I}^+} = \bigcup_{t \in W} A^{\mathcal{I}_t^+}$ and $A^{\mathcal{I}^-} = \bigcup_{t \in W} A^{\mathcal{I}_t^-}$. The $\mathcal{CALC}^2_\sim$-interpretation $C^{\mathcal{I}}$ of each $\mathcal{CALC}_\sim$-concept is given by $\bigcup_{t \in W} C^{\mathcal{I}_t^+}$. A $\mathcal{CALC}_\sim$-concept C (or a concept equation $C \equiv D$) is $\mathcal{CALC}^2_\sim$-satisfiable if there exists an $\mathcal{CALC}^2_\sim$-interpretation $\mathcal{I}$, called a $\mathcal{CALC}^2_\sim$-model of C (or $C \equiv D$), such that $C^{\mathcal{I}} \neq \emptyset$ (or $C^{\mathcal{I}} = D^{\mathcal{I}}$); otherwise, it is $\mathcal{CALC}^2_\sim$-unsatisfiable. In particular, if a $\mathcal{CALC}_\sim$-concept C is $\mathcal{CALC}^2_\sim$-satisfiable and the $\mathcal{CALC}^2_\sim$-model satisfies the contrary condition, then it is $\mathcal{CALC}^2_\sim$-satisfiable under the contrary condition. Otherwise, it is $\mathcal{CALC}^2_\sim$-unsatisfiable under the contrary condition. A concept equation $C \equiv D$ is $\mathcal{CALC}^2_\sim$-valid if every $\mathcal{CALC}^2_\sim$-interpretation $\mathcal{I}$ is a $\mathcal{CALC}^2_\sim$-model of $C \equiv D$. Contradictoriness and contrariness are preserved in an $\mathcal{CALC}^2_\sim$-interpretation of $\mathcal{CALC}_\sim$ if $\sim C^{\mathcal{I}} \subsetneq -C^{\mathcal{I}}$.

Theorem 4 (Contradictoriness and contrariness for $\mathcal{CALC}^2_\sim$).
Contradictoriness and contrariness are not preserved in some $\mathcal{CALC}^2_\sim$-interpretations that satisfy the contrary condition.

Table 1 shows the contradictoriness and contrariness for $\mathcal{ALC}_\sim$ and $\mathcal{CALC}_\sim$. The $\mathcal{CALC}_\sim$-concepts can be used to represent predicate denial and predicate term negation that capture conceptual models or describe a certain domain of interest; however, the contradictoriness and contrariness are not preserved in some $\mathcal{CALC}^2_\sim$-interpretations. For the $\mathcal{ALC}_\sim$-concepts, the contradictoriness and contrariness are preserved in every $\mathcal{ALC}^n_\sim$-interpretation since strong negation is suitably added to the classical description logic $\mathcal{ALC}$ without the undesirable equivalent $C \equiv \sim\neg C$ in the semantics. In the next section, the tableau-based satisfiability algorithm for $\mathcal{ALC}$ is extended to $\mathcal{ALC}_\sim$. This extension is based on the contradictoriness and contrariness for the $\mathcal{ALC}^n_\sim$-interpretations.

Additionally, we show that the replacement property holds for strongly equivalent $\mathcal{CALC}_\sim$-concepts. Let $C,\ D$ be $\mathcal{CALC}_\sim$-concepts. C and D are equivalent if, for every $\mathcal{CALC}^2_\sim$-interpretation $\mathcal{I}$, $C^{\mathcal{I}} = D^{\mathcal{I}}$. Let $C^{\mathcal{I}^+}$ denote $\bigcup_{t \in W} C^{\mathcal{I}_t^+}$ and $C^{\mathcal{I}^-}$ denote $\bigcup_{t \in W} C^{\mathcal{I}_t^-}$. C and D are strongly equivalent if, for every $\mathcal{CALC}^2_\sim$-interpretation $\mathcal{I}$, $C^{\mathcal{I}^+} = D^{\mathcal{I}^+}$ and $C^{\mathcal{I}^-} = D^{\mathcal{I}^-}$.

Table 1. Contradictoriness and contrariness for $\mathcal{ALC}_\sim$ and $\mathcal{CALC}_\sim$

DL Syntax	Semantics	Contradictoriness and contrariness
$\mathcal{ALC}_\sim$	$\mathcal{ALC}^2_\sim$	not preserved for every interpretation
	$\mathcal{ALC}^n_\sim$	preserved for every interpretation
$\mathcal{CALC}_\sim$	$\mathcal{CALC}^2_\sim$	not preserved for some interpretations

Theorem 5 (Replacement for $\mathcal{CALC}_\sim$). *Let $C,\ D$ be $\mathcal{CALC}_\sim$-concepts. If C and D are strongly equivalent, then E_C and E_D are (strongly) equivalent.*

4 Tableau-based algorithm for $\mathcal{ALC}_\sim$

We denote $rol(C)$ as the set of roles occurring in an $\mathcal{ALC}_\sim$-concept C. For instance, $rol(\neg\forall R_1.\exists R_2.C_1 \sqcup {\sim}C_2) = \{R_1, R_2\}$. To prove the soundness and completeness of the tableau-based satisfiability algorithm, a tableau for an $\mathcal{ALC}_\sim$-concept is created by adding conditions for the forms ${\sim}C$ and $({\sim}\neg)^i C$ to a tableau for an $\mathcal{ALC}$-concept (as in [6]).

Definition 4. *Let D be an $\mathcal{ALC}_\sim$-concept in the constructive normal negation form. A tableau T for D is a tuple (S, L, E), where S is a set of individuals, $L\colon S \to 2^{sub(D)}$ is a mapping from each individual into a set of concepts in $sub(D)$, and $E\colon rol(D) \to 2^{S\times S}$ is a mapping from each role into a set of pairs of individuals. There exists some $s_0 \in S$ such that $D \in L(s_0)$, and for all $s, t \in S$, the following conditions hold:*

1. *if $C \in L(s)$, then ${\sim}C, \neg C \notin L(s)$,*
2. *if $C_1 \sqcap C_2 \in L(s)$, then $C_1 \in L(s)$ and $C_2 \in L(s)$,*
3. *if $C_1 \sqcup C_2 \in L(s)$, then $C_1 \in L(s)$ or $C_2 \in L(s)$,*
4. *if $\forall R.C \in L(s)$ and $(s,t) \in E(R)$, then $C \in L(t)$,*
5. *if $\exists R.C \in L(s)$, then there exists $t \in S$ such that $(s,t) \in E(R)$ and $C \in L(t)$,*
6. *for every $i \in \omega$, if $({\sim}\neg)^{i+1}C \in L(s)$, then $({\sim}\neg)^i C \in L(s)$.*

In particular, it is called a C-tableau if the the following conditions hold:

7. *for every $i \in \omega$, there exists $s \in S$ such that $({\sim}\neg)^i C \in L(s)$ and $({\sim}\neg)^{i+1}C \notin L(s)$,*
8. *there exists $s \in S$ such that $C \notin L(s)$ and ${\sim}C \notin L(s)$.*

Conditions 1 and 6 reflect the $\mathcal{ALC}^n_\sim$-interpretation of $\mathcal{ALC}_\sim$-concepts combining classical and strong negations. Condition 1 states that $C \in L(s)$ implies ${\sim}C \notin L(s)$ (in addition to $\neg C \notin L(s)$) to satisfy the semantic condition $A^{\mathcal{I}_0^+} \cap A^{\mathcal{I}_0^-} = \emptyset$. Moreover, Condition 6 is imposed for the semantic conditions $A^{\mathcal{I}_{i+1}^+} \subseteq A^{\mathcal{I}_i^+}$ and $A^{\mathcal{I}_{i+1}^-} \subseteq A^{\mathcal{I}_i^-}$. For example, by Condition 6, if ${\sim}\neg{\sim}Happy \in L(s)$, then ${\sim}Happy \in L(s)$. In the corresponding semantics, if $d \in ({\sim}\neg{\sim}Happy)^{\mathcal{I}_0^+}$, then $d \in ({\sim}Happy)^{\mathcal{I}_1^+}$. Hence, by the condition $A^{\mathcal{I}_{i+1}^+} \subseteq A^{\mathcal{I}_i^+}$, we obtain $d \in ({\sim}Happy)^{\mathcal{I}_0^+}$. Conditions 7 and 8 correspond to the contrary condition for the $\mathcal{ALC}^n_\sim$-interpretation, i.e., $A^{\mathcal{I}_0^+} \cup A^{\mathcal{I}_0^-} \neq \Delta^{\mathcal{I}}$, $A^{\mathcal{I}_{i+1}^+} \subsetneq A^{\mathcal{I}_i^+}$, and $A^{\mathcal{I}_{i+1}^-} \subsetneq A^{\mathcal{I}_i^-}$. The next lemma shows the correspondence between the existence of a tableau for an $\mathcal{ALC}_\sim$-concept and its satisfiability.

$\sqcap$**-rule:** $L(x) = L(x) \cup \{C_1, C_2\}$
if $C_1 \sqcap C_2 \in L(x)$ and $\{C_1, C_2\} \not\subseteq L(x)$

$\sqcup$**-rule:** $L(x) = L(x) \cup \{C_1\}$ or $L(x) = L(x) \cup \{C_2\}$
if $C_1 \sqcup C_2 \in L(x)$ and $\{C_1, C_2\} \cap L(x) = \emptyset$

$\forall$**-rule:** $L(y) = L(y) \cup \{C\}$
if $\forall R.C \in L(x)$, $(x, y) \in E(R)$ and $C \notin L(y)$

$\exists$**-rule:** $S = S \cup \{y\}$ with $y \notin S$, $E(R) = E(R) \cup \{(x, y)\}$ and $L(y) = \{C\}$
if $\exists R.C \in L(x)$ and $\{z \mid (x, z) \in E(R), C \in L(z)\} = \emptyset$

$(\sim\neg)^i$**-rule 1:** $L(x) = L(x) \cup \{(\sim\neg)^i C\}$
if $(\sim\neg)^{i+1} C \in L(x)$ and $(\sim\neg)^i C \notin L(x)$

$(\sim\neg)^i$**-rule 2:** $S = S \cup \{y\}$ with $y \notin S$ and $L(y) = \{(\sim\neg)^i C\}$
if $(\sim\neg)^i C \in L(x)$ and
there exists no $z \in S$ such that $(\sim\neg)^{i+1} C \notin L(z)$ and $(\sim\neg)^i C \in L(z)$

$\sim$**-rule:** $S = S \cup \{y\}$ with $y \notin S$ and $L(y) = \emptyset$
if $A \in L(x)$ or $\sim A \in L(x)$ and
there exists no $z \in S$ such that $\{A, \sim A\} \cap L(z) = \emptyset$.

Fig. 3. Completion rules for $\mathcal{ALC}_\sim$-concepts

Lemma 3. *Let D be an $\mathcal{ALC}_\sim$-concept. There exists a tableau for D if and only if it is $\mathcal{ALC}^n_\sim$-satisfiable. In particular, there exists a C-tableau for D if and only if it is $\mathcal{ALC}^n_\sim$-satisfiable under the contrary condition.*

Note that while every $\mathcal{ALC}^n_\sim$-interpretation $\mathcal{I}$ consists of infinite interpretation functions $\cdot^{\mathcal{I}_i^+}$ and $\cdot^{\mathcal{I}_i^-}$ for $i \in \omega$, the tableau corresponding to an $\mathcal{ALC}^n_\sim$-model of an $\mathcal{ALC}_\sim$-concept is finitely defined. Each $\mathcal{ALC}_\sim$-concept can be satisfied by finite interpretation functions because the number of connectives occurring in it is finite. For example, $(\sim\neg)^m A$ can be satisfied by the maximum $2m+1$ of interpretation functions $\cdot^{\mathcal{I}_0^+}, \ldots, \cdot^{\mathcal{I}_m^+}, \cdot^{\mathcal{I}_0^-}, \ldots, \cdot^{\mathcal{I}_{m-1}^-}$. Lemma 3 indicates that given a tableau for an $\mathcal{ALC}_\sim$-concept D, we can define an $\mathcal{ALC}^n_\sim$-interpretation $\mathcal{I}$ satisfying it (i.e., its $\mathcal{ALC}^n_\sim$-model). The model is constructed in such a manner that for every constructive double negation $(\sim\neg)^i A$ (resp. $\sim(\neg\sim)^i A$) in $sub(D)$, $A^{\mathcal{I}_i^+}$ (resp. $A^{\mathcal{I}_i^-}$) is defined by the set of individuals $\{s \mid (\sim\neg)^i A \in L(s)\}$ (resp. $\{s \mid \sim(\neg\sim)^i A \in L(s)\}$). Thus, the finiteness of a tableau for an $\mathcal{ALC}_\sim$-concept leads to the termination of its satisfiability algorithm which we will present.

In order to determine the satisfiability of $\mathcal{ALC}_\sim$-concepts, the tableau-based algorithm for $\mathcal{ALC}$ will be extended by introducing three new completion rules ($(\sim\neg)^i$-rule 1, $(\sim\neg)^i$-rule 2, and $\sim$-rule) and clash forms with respect to strong negation and constructive double negation. In Fig.3, the completion rules for $\mathcal{ALC}_\sim$-concepts are presented (as in [4, 13]). $(\sim\neg)^i$-rule 1 is applied to $\mathcal{ALC}_\sim$-concepts of the forms $(\sim\neg)^i A$ and $\sim(\neg\sim)^i A$. $(\sim\neg)^i$-rule 2 and $\sim$-rule introduce new variables if there exists no $z \in S$ such that $(\sim\neg)^{i+1} C \notin L(z)$ and $(\sim\neg)^i C \in L(z)$; or $\{A, \sim A\} \cap L(z) = \emptyset$.

Remark. The algorithm has to recognize additional clash forms besides $\{A, \neg A\}$ and $\{\bot\}$. That is, $L(x)$ contains a clash if it contains $\{C_1, \neg C_1\}$, $\{C_2, \sim C_2\}$, or $\{\bot\}$, where

C_1 is of the form $(\sim\neg)^i A$ or $\sim(\neg\sim)^i A$ and C_2 is of the form $(\neg\sim)^i A$ or $\neg(\sim\neg)^i A$. For example, if $\{\neg\sim\neg A_1, \sim\neg\sim\neg A_1\} \subseteq L(x_1)$, then it contains a clash.

We present a tableau-based satisfiability algorithm for $\mathcal{ALC}_\sim$. Given an $\mathcal{ALC}_\sim$-concept D, the following procedure constructs a forest $ST = (S, E_{rol(D)} \cup E_\sim \cup E_{(\sim\neg)^i}, x_0)$ for D, where S is a set of individuals, each node $x \in S$ is labeled as $L(x)$, $E_{rol(D)} = \{(x,y) \in E(R) \mid R \in rol(D)\}$ (each edge $(x,y) \in E(R)$ is labeled as R), $(x,y) \in E_\sim \Leftrightarrow y$ is introduced for $A \in L(x)$ or $\sim A \in L(x)$ in $\sim$-rule, $(x,y) \in E_{(\sim\neg)^i}$ $\Leftrightarrow y$ is introduced for $(\sim\neg)^i C \in L(x)$ in $(\sim\neg)^i$-rule 2, and x_0 is the root. First, set the initial forest $ST = (\{x_0\}, \emptyset, x_0)$, where $S = \{x_0\}$, $L(x_0) = \{D\}$, $E(R) = \emptyset$ for all $R \in rol(D)$, and $E_\sim = E_{(\sim\neg)^i} = \emptyset$. Then, apply completion rules in Fig.3 to ST until none of the rules are applicable. A forest ST is called complete if any completion rule is not applicable to it. If there is a clash-free complete forest ST, then return "satisfiable," and otherwise return "unsatisfiable."

We show the correctness of the tableau-based satisfiability algorithm under the contrary condition (soundness, completeness and termination)[3] and the complexity of the satisfiability problem. We sketch the behavior of the new completion rules in the algorithm. Unlike the other completion rules, each application of the new completion rules does not subdivide a concept. However, since $(\sim\neg)^i$-rules 1 and 2 add only a subconcept to each node, $\sim$-rule creates an empty node, and the number of variables introduced in $(\sim\neg)^i$-rule 2 and $\sim$-rule is bounded by polynomial, the termination can be established.

Theorem 6 (Satisfiability under the contrary condition).
Let D be an $\mathcal{ALC}_\sim$-concept. The following statements hold:

1. *The tableau-based algorithm terminates.*
2. *The tableau-based algorithm constructs a clash-free complete forest for an $\mathcal{ALC}_\sim$-concept D if and only if D is $\mathcal{ALC}^n_\sim$-satisfiable under the contrary condition.*
3. *Satisfiability of $\mathcal{ALC}_\sim$-concepts is PSPACE-complete.*

5 Conclusion

We have presented an extended description logic $\mathcal{ALC}_\sim$ that incorporates classical negation and strong negation for representing contraries, contradictories, and subcontraries between concepts. The important specification of the description logic is that strong negation is added to the classical description logic $\mathcal{ALC}$ and the property of contradictoriness and contrariness holds for every interpretation. The two negations are adequately characterized by $\mathcal{ALC}^n_\sim$-interpretations, unlike $\mathcal{ALC}^2_\sim$- and $\mathcal{CALC}^2_\sim$-interpretations. Technically, the semantics of strong negation is adapted to the oppositions in the philosophical study of negation [7]. Furthermore, we have developed a satisfiability algorithm for $\mathcal{ALC}_\sim$ that is extended to add three new completion rules to the tableau-based algorithm for $\mathcal{ALC}$ [13]. It constructs an $\mathcal{ALC}^n_\sim$-model satisfying the contrary condition, in which a constructive normal negation form and various

[3] The tableau-based satisfiability algorithm is complete in the class of $\mathcal{ALC}^n_\sim$-interpretations where $(\sim\neg)^i$-rule 2 and $\sim$-rule are used to construct a model satisfying the contrary condition.

clash forms are defined to treat complex negative concepts. The description logic provides a decidable fragment (precisely, PSPACE-complete) of classical first-order logic with classical negation and strong negation (but not constructive description logic with Heyting negation and strong negation).

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