Description Logics with Contraries, Contradictories, and Subcontraries[1]

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Abstract Several constructive description logics,[12)] in which classical negation was replaced by strong negation as a component to treat negative atomic information have been proposed as intuitionistic variants of description logics. For conceptual representation, strong negation alone and in a combination with classical negation seems to be useful and necessary due to their respective predicate denial (e.g., not happy) and predicate term negation (e.g., unhappy) properties. In this paper, we propose an alternative description logic $\mathcal{ALC}^n_\sim$ with classical negation and strong negation. We adhere in particular to the notions of contraries, contradictories, and subcontraries (as discussed in [6)]), generated from conceivable statement types using predicate denial and predicate term negation. To capture these notions, our formalization includes a semantics that suitably interprets various combinations of classical negation and strong negation. We show that our semantics preserves contradictoriness and contrariness for $\mathcal{ALC}^n_\sim$-concepts, but the semantics of constructive description logic $\mathcal{CALC}^2_\sim$ with Heyting negation and strong negation cannot preserve the property for $\mathcal{CALC}^2_\sim$-concepts.

Keywords: *negative information, constructive description logic, strong negation, terminological knowledge representation*

§1 Introduction

[*1] This is an extended version of the paper.[8)]

Negative information plays an important role in knowledge representation and reasoning (cf. [19, 10, 21]). Classical negation $\neg F$ represents the negation of a statement F, but a strong negation $\sim F$ may be more suitable for expressing explicit negative information (or negative facts). In other words, $\sim F$ indicates information that is directly opposite and exclusive to F rather than its complementary negation. Therefore, the law of contradiction $\neg(F \wedge \sim F)$ holds, but the law of excluded middle $F \vee \sim F$ does not hold.[11, 1, 19] For example, given the formula $rich(x)$ that represents "x is rich," the antonymous formula $poor(x)$ is defined by the strong negation $\sim rich(x)$, and not by the classical negation $\neg rich(x)$. Thus, we can logically recognize the inconsistency of $rich(x)$ and $\sim rich(x)$ (as $poor(x)$), and, because $rich(x) \vee \sim rich(x)$ is not valid, we can represent "x is neither rich nor poor" as $\neg rich(x) \wedge \neg\sim rich(x)$.

In conceptual knowledge representation, negative concepts are necessary because, when defining certain concepts, people may wish to use the complement of one concept to define another. For example, using description logics,[2] the concept *Bachelor* can be defined by using the complement of $\exists hasWife.Woman$ as a specialization of the concept *Man*.

$$Bacelor \equiv \neg(\exists hasWife.Woman) \sqcap Man$$

This subconcept $\exists hasWife.Woman$ describes an individual who has a wife; and further, that the wife is a woman.

In addition to the complement corresponding to classical negation, it is conceivable that many concept names implicitly contain negative meanings as lexical negations.[*1] Negative affixes (e.g., in-, il-, un-, and non-) generate negative concepts such as *Unfix*, *Illogical*, *Incoherent*, *Inactive*, *Impolite*, *Non-selfish*, etc. Obviously, positive concepts are obtained by deleting their affixes. A lexicon with a negative meaning implies the negation of a positive concept. For instance, the concepts *Doubt*, *Deny*, *Dissuade*, and *Forget* imply the negation of the concepts *Believe*, *Say*, *Persuade*, and *Remember*, respectively. Unlike the complements of positive concepts, these lexical negations correspond to strong negation rather than classical negation. However, standard description logics do not support strong negation, which results in two problems. The first is that if we try to use the negative concepts *Unhappy*, *Displeased*, and *Unsatisfied* in the concept hierarchy shown in Figure 1, the following axioms in the description logic $\mathcal{ALC}$ will have to be supplemented to characterize their negative meanings.

[*1] See [13, 10].

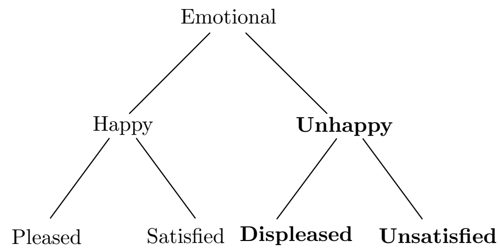


Fig. 1 A Concept Hierarchy with Negative Names

Problem 1: Redundant expressions in $\mathcal{ALC}$

$$
\begin{aligned}
&\mathit{Unhappy} \sqsubseteq \neg \mathit{Happy} \sqcap \mathit{Emotional}\\
&\mathit{Happy} \sqcup \mathit{Unhappy} \not\equiv \top\\
&\mathit{Displeased} \sqsubseteq \neg \mathit{Pleased} \sqcap \mathit{Unhappy}\\
&\mathit{Pleased} \sqcup \mathit{Displeased} \not\equiv \top\\
&\mathit{Unsatisfied} \sqsubseteq \neg \mathit{Satisfied} \sqcap \mathit{Unhappy}\\
&\mathit{Satisfied} \sqcup \mathit{Unsatisfied} \not\equiv \top
\end{aligned}
$$

The second axiom is the negation of a concept equivalence. Therefore, it cannot be expressed in the description logic knowledge base. Instead of checking for this equivalence, we must check whether the negation of the disjunction is not empty, that is, satisfiable. Therefore, without adding these axioms we use simpler descriptions of strong negation $\sim\mathit{Happy}$, $\sim\mathit{Pleased}$, and $\sim\mathit{Satisfied}$ rather than their translation into the $\mathcal{ALC}$.

The second problem is that concept designers may use complex concepts, including strong negation, in a manner that makes it difficult to express the concepts in the description logic $\mathcal{ALC}$.

Problem 2: How to express the following complex concepts in $\mathcal{ALC}$?

$$
\begin{aligned}
&\mathit{Happy} \equiv \neg \mathit{Sad} \sqcap \mathit{Pleased}\\
&{\sim}\mathit{Happy} \equiv {\sim}(\neg \mathit{Sad} \sqcap \mathit{Pleased})\\
&\neg{\sim}\mathit{Happy} \equiv \neg{\sim}(\neg \mathit{Sad} \sqcap \mathit{Pleased})\\
&{\sim}\neg{\sim}\mathit{Happy} \equiv {\sim}\neg{\sim}(\neg \mathit{Sad} \sqcap \mathit{Pleased})\\
&\qquad\vdots
\end{aligned}
$$

Introducing a new operation ($\sim$) to denote strong negation solves the problem. To support the negation, the operation should be adequately formalized in the concept language.

Odintsove and Wansing[12] proposed several constructive description logics as intuitionistic variants of description logics in which classical negation was replaced by strong negation as a component to treat negative atomic information. In contrast, since basic description logics correspond to a subclass of *classical* first-order logic (i.e., they have classical negation), negative concepts are expressed by classical negation. Due to the different features of negative information, complex negative statements using strong negation and classical negation can be usefully employed in conceptual representation. In the philosophical study of negation, there is a distinction between predicate denial (e.g., not happy) and predicate term negation (e.g., unhappy).[6, 21] Moreover, conceivable statement types using predicate denial and predicate term negation give rise to opposition between affirmation and negation (contraries, contradictories, and subcontraries).[6] To deal logically with such negation types in concept languages, it is required that the difference between classical negation and strong negation must be embedded into the syntax and semantics of description logics.

In this paper, we propose a description logic $\mathcal{ALC}^n_\sim$ extended with strong negation, that is, an extension of the basic description logic $\mathcal{ALC}$. The following are our primary results. First, we present a semantics of $\mathcal{ALC}^n_\sim$ that adheres to oppositions – contraries, contradictories, and subcontraries in concept languages. We point out that the conventional semantics of strong negation[1] yields the undesirable equivalence $C \equiv {\sim}\neg C$ as opposed to our proposed semantics, where there is an implication between C and ${\sim}\neg C$ instead of an equivalence. Secondly, we show that our semantics preserves the property of contradictoriness and contrariness that there exists an element such that it belongs to the contradictory negation $\neg C$ but not to the contrary negation ${\sim}C$. The property characterizes a distinction between the complement of a concept and the contrary opposition of the concept such as $\neg Rich$ and ${\sim}Rich$ by imposing it on the semantics. Even if a user has a model where there is no instance of $\neg Rich \sqcap \neg{\sim}Rich$, consistency checking requires the property. This is similar to a case in which, if consistency is checked, every concept must contain an element, but a user may have a model where some concepts do not include any element. When Heyting negation and classical negation are being used in constructive description logics, the property cannot be preserved for some interpretations. This motivates us to redesign a new semantics for contradictory negation and contrary negation. Based on our semantics, we will develop a tableau-based algorithm for concept satisfiability in $\mathcal{ALC}^n_\sim$ and show the correctness – soundness, completeness, and termination. Although three new completion rules are used to deal with the two negations, the extended description logic $\mathcal{ALC}^n_\sim$ does not increase the complexity of the satisfiability problem in the basic description logic $\mathcal{ALC}$.

§2 Preliminaries

This section describes the syntax and semantics of the basic description logic $\mathcal{ALC}$ and briefly introduces the notions of contradictories, contraries, and strong negation.

2.1 Classical description logic: $\mathcal{ALC}$

The basic description logic $\mathcal{ALC}$[16)] contains a set $\mathbf{C}$ of concept names A including $\bot$ and $\top$, a set $\mathbf{R}$ of role names R, and a set $\mathbf{I}$ of individual names a. The bottom concept $\bot$ and the universal concept $\top$ represent the empty set and the set of individuals, respectively. The concepts of the language (called $\mathcal{ALC}$-concepts) are constructed by concept names A; role names R; the connectives $\sqcap$, $\sqcup$, and $\neg$ (classical negation); and the quantifiers $\forall$ and $\exists$. Every concept name $A \in \mathbf{C}$ is an $\mathcal{ALC}$-concept. If R is a role name and C, D are $\mathcal{ALC}$-concepts, then $\neg C$, $C \sqcap D$, $C \sqcup D$, $\forall R.C$, and $\exists R.C$ are $\mathcal{ALC}$-concepts. Concepts are used to represent classes as sets of individuals, and roles are used to specify their properties and attributes. Let $Male, Human, Girl$, and Boy be concept names, and let *has-child* be a role name. For example, the $\mathcal{ALC}$-concept $\neg Girl$ (negation of a concept) expresses "individuals who are not girls." The $\mathcal{ALC}$-concepts $Male \sqcap Human$ (intersection of concepts) and $Boy \sqcup Girl$ (union of concepts) represent "individuals who are male and female" and "individuals who are boys or girls," respectively. Moreover, $\exists$*has-child.Male* (existential quantification) represents "individuals who have sons" and $\forall$*has-child.Male* (universal quantification) expresses "individuals whose children are all male."

An $\mathcal{ALC}$-interpretation $\mathcal{I}$ is a tuple $(\Delta^\mathcal{I}, \cdot^\mathcal{I})$, where $\Delta^\mathcal{I}$ is a non-empty set and $\cdot^\mathcal{I}$ is an interpretation function ($A^\mathcal{I} \subseteq \Delta^\mathcal{I}$, $R^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$, and $a^\mathcal{I} \in \Delta^\mathcal{I}$) such that:

$$\bot^\mathcal{I} = \emptyset \text{ and } \top^\mathcal{I} = \Delta^\mathcal{I}.$$

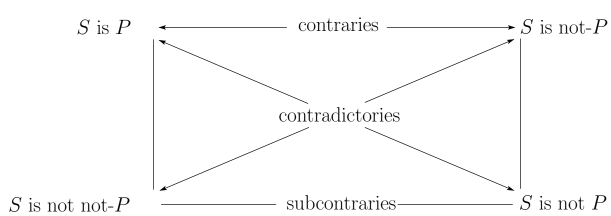
The interpretation function $\cdot^{\mathcal{I}}$ is expanded to $\mathcal{ALC}$-concepts as follows:

$$\begin{aligned}
(\neg C)^{\mathcal{I}} &= \Delta^{\mathcal{I}} - C^{\mathcal{I}} \\
(C \sqcap D)^{\mathcal{I}} &= C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \sqcup D)^{\mathcal{I}} &= C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(\forall R.C)^{\mathcal{I}} &= \{d_1 \in \Delta^{\mathcal{I}} \mid \forall d_2[(d_1, d_2) \in R^{\mathcal{I}} \text{ implies } d_2 \in C^{\mathcal{I}}]\} \\
(\exists R.C)^{\mathcal{I}} &= \{d_1 \in \Delta^{\mathcal{I}} \mid \exists d_2[(d_1, d_2) \in R^{\mathcal{I}} \text{ and } d_2 \in C^{\mathcal{I}}]\}
\end{aligned}$$

2.2 Contradictories, contraries, and strong negation

The philosophical study of negation makes a distinction between predicate denial and predicate term negation. As Horn explained in his paper[6], "The predicate denial—A is not B—has for Aristotle the appropriate semantics for contradictory negation; it is true if and only if the corresponding affirmation—A is B—is false. ... alongside ordinary predicate denial, Aristotle acknowledges the existence of term negation, in which a negative predicate term (not-ill) is affirmed of a subject."

Conceivable statement types using predicate denial and predicate term negation derive an opposition between affirmation and negation, such as contraries, contradictories, and subcontraries, as shown in the following figure.



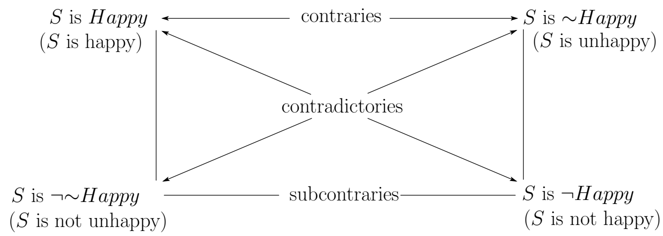
A proposition 'S is P' derives its contradictory opposition 'S is not P' by predicate denial and its contrary opposition 'S is not-P' by term negation. The statement 'S is not not-P' is the contradictory opposition of 'S is not-P' as an affirmation of 'S is P.'

In intuitionistic logic, Heyting negation and strong negation exist as methods of dealing with the oppositions. In general, strong negation has been formalized as a constructive negation of intuitionistic logic. Thomason[17] proposed the semantics of intuitionistic first-order logic only with strong negation, Akama[1] formalized intuitionistic logic with Heyting negation and strong negation, Wagner[18] and Herre et al.[4] defined weak negation and strong negation in the logic they developed, and Pearce and Wagner[14] proposed logic programming with strong negation that was regarded as a subsystem of intuitionistic logic.

Here, we observe the properties of Heyting negation ($-$) and strong negation ($\sim$). Let F be a formula. The law of double negation $--F \leftrightarrow F$ and the law of the excluded middle $-F \vee F$ do not hold for Heyting negation, but the law of double negation $\sim\sim F \leftrightarrow F$ is valid for strong negation. The combinations of these negations lead to the valid axiom $\sim - F \leftrightarrow F$. Hence, the contradictory F and $-F$ can be replaced with $\sim - F$ and $-F$ by the equivalence $\sim - F \leftrightarrow F$, but they should be recognized as a contrary. This is in partial disagreement with the abovementioned oppositions in which contradictories and contraries are defined differently. With regard to these features, Heyting negation and strong negation using the conventional semantics in intuitionistic logic[17, 1, 4] do not satisfy our requirements. Thus, we need to model contradictory negation and contrary negation in description logics and compare them with the constructive description logics proposed by Odintsove and Wansing[12].

§3 Contradictories and contraries between concepts

Strong negation can be used to describe conceptual oppositions (e.g., *Happy* and *Unhappy*) in description logics. For example, let us denote by *Happy*, $\neg$*Happy* (classical negation), and $\sim$*Happy* (strong negation) the concepts "individuals that are happy," "individuals that are not happy," and "individuals that are unhappy," respectively. We can then construct the complex concepts – $\exists$*has-child*.$\neg$*Happy* as "parents who have children that are not happy," $\exists$*has-child*.$\sim$*Happy* as "parents who have unhappy children," and $(\neg \mathit{Happy} \sqcap \neg{\sim}\mathit{Happy}) \sqcap \mathit{Person}$ as "persons who are neither happy nor unhappy." Syntactically, these allow us to express concepts composed of various combinations of classical negation and strong negation, e.g., $\sim\neg C$ and $\sim\neg\sim C$. As discussed in [15], the two negations represent the following oppositions between affirmation and negation (which Horn[6] renders):



In our conceptual explanation of them, the contraries (*Happy* and $\sim$*Happy*) imply that neither concept can contain an identical element. The contradictories (*Happy* and $\neg$*Happy*) imply that one concept must contain an element when it does not belong to the other. The subcontraries ($\neg\sim$*Happy* and $\neg$*Happy*) imply that every element belongs to either of the concepts. To apply the oppositions to any given DL-concepts, we must generalize them as follows:[*2]

Contraries:	$(\neg\sim)^i A$ and $\sim(\neg\sim)^i A$
	$\neg(\sim\neg)^i A$ and $(\sim\neg)^{i+1} A$
Contradictories:	$(\sim\neg)^i A$ and $\neg(\sim\neg)^i A$
	$\sim(\neg\sim)^i A$ and $(\neg\sim)^{i+1} A$
Subcontraries:	$(\neg\sim)^{i+1} A$ and $\neg(\neg\sim)^i A$
	$\neg(\sim\neg)^{i+1} A$ and $(\sim\neg)^i A$

where A is a concept name (i.e., an atomic concept). In the above, if X and Y are contraries (contradictories or subcontraries), X is a contrary (contradictory or subcontrary) of Y and Y is a contrary (contradictory or subcontrary) of X. In reasoning algorithms for description logics with the two negations, contraries and contradictories will be taken as a criterion for checking inconsistent pairs of DL-concepts.

While intuitionistic logic and strong negation allow us to represent term negation and predicate denial, we would like to propose a description logic such that:

1. It contains classical negation and strong negation, since $\mathcal{ALC}$ is based on classical logic.
2. It fulfills the property that contradictoriness and contrariness are preserved for every interpretation.

[*2] $(\sim\neg)^i$ (resp. $(\neg\sim)^i$) denotes a chain of length i of $\sim\neg$ (resp. $\neg\sim$).

Let C be a DL-concept and let $C(x)$ denote a first-order formula in one free variable that corresponds to C.[*3] To incorporate strong negation into classical first-order logic and to remove the equivalence $\sim\neg C(x) \leftrightarrow C(x)$, we have improved the semantics by capturing the following properties.[9)] The law of double negation $\neg\neg C(x) \leftrightarrow C(x)$ and the law of the excluded middle $\neg C(x) \vee C(x)$ hold for classical negation, and the equivalence $\sim\neg C(x) \leftrightarrow C(x)$ is not valid. In conceptual representation based on this semantics, the strong negation $\sim C$ of a concept C is partial and exclusive to its affirmative expression C. The partiality of strong negation entails the existence of information that is neither affirmative nor strongly negative, i.e., $\neg C \sqcap \neg{\sim}C \not\equiv \bot$. In contrast, the classical negation $\neg C$ is complementary and exclusive to its affirmative expression C. Hence, the disjunction of affirmation and its classical negation expresses the set of all individuals, i.e., $C \sqcup \neg C \equiv \top$. Additionally, the simple double negations $\neg\neg C$ and $\sim\sim C$ are interpreted to be equivalent to the affirmation C. We can constructively define the complex double negations $\sim\neg C$ and $\neg{\sim}C$ without losing the features of the two negations by refining the conventional semantics. If we strongly deny the classical negation $\neg C$, then the double negation $\sim\neg C$ (called *constructive double negation*) must be partial and exclusive to $\neg C$. If we express the classical negation of a strong negation $\sim C$, then the double negation $\neg{\sim}C$ (called *weak double negation*) must be complementary and exclusive to $\sim C$.

§4 Strong negation in description logics

In this section, we define description logics with classical negation and strong negation and analyze the property of contradictoriness and contrariness for the proposed logic. In addition, we define a constructive description logic obtained by including Heyting negation and strong negation.

4.1 Description logics with classical negation and strong negation: $\mathcal{ALC}^2_\sim$ and $\mathcal{ALC}^n_\sim$

The description logic $\mathcal{ALC}^2_\sim$ (as an extension of $\mathcal{ALC}$[16)]) is based upon a set $\mathbf{C}$ of concept names A (including $\top$ and $\bot$), a set $\mathbf{R}$ of role names R, and a set $\mathbf{I}$ of individual names a. The concepts of the language (called $\mathcal{ALC}^2_\sim$-concepts) are constructed by concept names A; role names R; the connectives $\sqcap$, $\sqcup$, $\neg$ (classical negation), and $\sim$ (strong negation); and the quantifiers $\forall$ and

[*3] Each DL-concept C is interpreted as a subset of the universe that indicates the set of instances of the concept. Let $p_C(x)$ (simply denoted $C(x)$) be a unary predicate formula indexed by the concept name where p_C is interpreted by the same set.

$\exists$. Every concept name $A \in \mathbf{C}$ is an $\mathcal{ALC}^2_\sim$-concept. If R is a role name and C, D are $\mathcal{ALC}^2_\sim$-concepts, then $\neg C$, $\sim C$, $C \sqcap D$, $C \sqcup D$, $\forall R.C$, and $\exists R.C$ are $\mathcal{ALC}^2_\sim$-concepts.

We denote as $sub(C)$ the set of subconcepts of an $\mathcal{ALC}^2_\sim$-concept C. Let X be a sequence of classical negation $\neg$ and strong negation $\sim$. We denote $(X)^n$ as a chain of length n of X. For instance, $\sim(\neg\sim)^2 C_1$ and $(\sim\neg)^0 C_2$ denote $\sim\neg\sim\neg\sim C_1$ and C_2, respectively. Next, we define an interpretation of $\mathcal{ALC}^2_\sim$-concepts (called an $\mathcal{ALC}^2_\sim$-interpretation) by using the conventional semantics of strong negation.

Definition 4.1

An $\mathcal{ALC}^2_\sim$-interpretation $\mathcal{I}$ is a tuple $(\Delta^\mathcal{I}, \cdot^{\mathcal{I}^+}, \cdot^{\mathcal{I}^-})$, where $\Delta^\mathcal{I}$ is a non-empty set and $\cdot^{\mathcal{I}^+}$ and $\cdot^{\mathcal{I}^-}$ are interpretation functions ($A^{\mathcal{I}^+}$, $A^{\mathcal{I}^-} \subseteq \Delta^\mathcal{I}$, $R^{\mathcal{I}^+} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$, and $a^{\mathcal{I}^+} \in \Delta^\mathcal{I}$) such that:

1. $\bot^{\mathcal{I}^+} = \emptyset$ and $\top^{\mathcal{I}^+} = \Delta^\mathcal{I}$,
2. $A^{\mathcal{I}^+} \cap A^{\mathcal{I}^-} = \emptyset$.

The interpretation functions $\cdot^{\mathcal{I}^+}$ and $\cdot^{\mathcal{I}^-}$ are expanded to $\mathcal{ALC}^2_\sim$-concepts as follows:

$$\begin{aligned}
(\neg C)^{\mathcal{I}^+} &= \Delta^\mathcal{I} \backslash C^{\mathcal{I}^+} & (\sim C)^{\mathcal{I}^+} &= C^{\mathcal{I}^-} \\
(C \sqcap D)^{\mathcal{I}^+} &= C^{\mathcal{I}^+} \cap D^{\mathcal{I}^+} & (C \sqcup D)^{\mathcal{I}^+} &= C^{\mathcal{I}^+} \cup D^{\mathcal{I}^+} \\
(\forall R.C)^{\mathcal{I}^+} &= \{d_1 \in \Delta^\mathcal{I} \mid \forall d_2[(d_1, d_2) \in R^{\mathcal{I}^+} \to d_2 \in C^{\mathcal{I}^+}]\} \\
(\exists R.C)^{\mathcal{I}^+} &= \{d_1 \in \Delta^\mathcal{I} \mid \exists d_2[(d_1, d_2) \in R^{\mathcal{I}^+} \wedge d_2 \in C^{\mathcal{I}^+}]\} \\
(\neg C)^{\mathcal{I}^-} &= C^{\mathcal{I}^+} & (\sim C)^{\mathcal{I}^-} &= C^{\mathcal{I}^+} \\
(C \sqcap D)^{\mathcal{I}^-} &= C^{\mathcal{I}^-} \cup D^{\mathcal{I}^-} & (C \sqcup D)^{\mathcal{I}^-} &= C^{\mathcal{I}^-} \cap D^{\mathcal{I}^-} \\
(\forall R.C)^{\mathcal{I}^-} &= \{d_1 \in \Delta^\mathcal{I} \mid \exists d_2[(d_1, d_2) \in R^{\mathcal{I}^+} \wedge d_2 \in C^{\mathcal{I}^-}]\} \\
(\exists R.C)^{\mathcal{I}^-} &= \{d_1 \in \Delta^\mathcal{I} \mid \forall d_2[(d_1, d_2) \in R^{\mathcal{I}^+} \to d_2 \in C^{\mathcal{I}^-}]\}
\end{aligned}$$

An $\mathcal{ALC}^2_\sim$-interpretation $\mathcal{I}$ satisfies the contrary condition if for all concept names A, $A^{\mathcal{I}^+} \cup A^{\mathcal{I}^-} \neq \Delta^\mathcal{I}$. The $\mathcal{ALC}^2_\sim$-interpretation is defined by the two interpretation functions $\cdot^{\mathcal{I}^+}$ and $\cdot^{\mathcal{I}^-}$, but it causes an undesirable equation $C \equiv \sim\neg C$, i.e., $C^{\mathcal{I}^+} = (\sim\neg C)^{\mathcal{I}^+}$. Contradictoriness and contrariness are preserved in an $\mathcal{ALC}^2_\sim$-interpretation $\mathcal{I}$ of $\mathcal{ALC}^2_\sim$ if $\sim C^\mathcal{I} \subsetneq \neg C^\mathcal{I}$. The interpretation results in the following negative property:

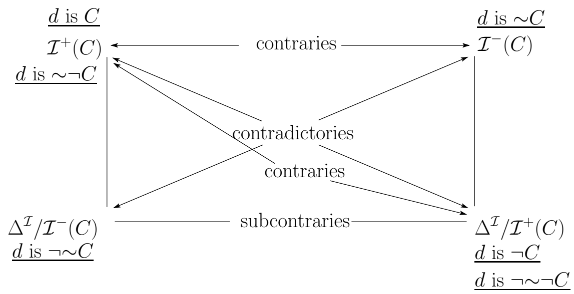


Fig. 2 Oppositions in $\mathcal{ALC}^2_{\sim}$-interpretations

Theorem 4.1 (Contradictoriness and contrariness for $\mathcal{ALC}^2_{\sim}$)
Contradictoriness and contrariness are not preserved in any $\mathcal{ALC}^2_{\sim}$-interpretation.

Proof Let $\mathcal{I}$ be any $\mathcal{ALC}^2_{\sim}$-interpretation. By definition, $A^{\mathcal{I}^+} = (\neg\neg A)^{\mathcal{I}^+} = ({\sim}\neg A)^{\mathcal{I}^+}$. Therefore, contradictoriness and contrariness are not preserved in every $\mathcal{ALC}^2_{\sim}$-interpretation. ∎

Subsequently, the concepts of the language (called $\mathcal{ALC}^n_{\sim}$-concepts) are constructed by concept names A; role names R; the connectives $\sqcap$, $\sqcup$, $\neg$ (classical negation), and $\sim$ (strong negation); and the quantifiers $\forall$ and $\exists$. Every concept name $A \in \mathbf{C}$ is an $\mathcal{ALC}^n_{\sim}$-concept. If R is a role name and C, D are $\mathcal{ALC}^n_{\sim}$-concepts, then $\neg C$, ${\sim}C$, $C \sqcap D$, $C \sqcup D$, $\forall R.C$, and $\exists R.C$ are $\mathcal{ALC}^n_{\sim}$-concepts. We define an interpretation of $\mathcal{ALC}^n_{\sim}$-concepts (called an $\mathcal{ALC}^n_{\sim}$-interpretation), which is based on the semantics[9)] obtained by improving Akama's semantics.[1)]

Definition 4.2
An $\mathcal{ALC}^n_{\sim}$-interpretation $\mathcal{I}$ is a tuple $(\Delta^{\mathcal{I}}, \{\cdot^{\mathcal{I}_i^+} \mid i \in \omega^{*4}\}, \{\cdot^{\mathcal{I}_i^-} \mid i \in \omega\})$, where $\Delta^{\mathcal{I}}$ is a non-empty set and $\cdot^{\mathcal{I}_i^+}$ and $\cdot^{\mathcal{I}_i^-}$ are interpretation functions ($A^{\mathcal{I}_i^+}$, $A^{\mathcal{I}_i^-} \subseteq \Delta^{\mathcal{I}}$, $R^{\mathcal{I}_0^+} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, and $a^{\mathcal{I}_0^+} \in \Delta^{\mathcal{I}}$), such that:

1. $\bot^{\mathcal{I}_0^+} = \emptyset$ and $\top^{\mathcal{I}_0^+} = \Delta^{\mathcal{I}}$,
2. $A^{\mathcal{I}_0^+} \cap A^{\mathcal{I}_0^-} = \emptyset$,
3. $A^{\mathcal{I}_{i+1}^+} \subseteq A^{\mathcal{I}_i^+}$ and $A^{\mathcal{I}_{i+1}^-} \subseteq A^{\mathcal{I}_i^-}$.

The interpretation functions $\cdot^{\mathcal{I}_i^+}$ and $\cdot^{\mathcal{I}_i^-}$ are expanded to $\mathcal{ALC}^n_{\sim}$-concepts as

[*4] The symbol ω denotes the set of natural numbers. Thus, $\{\cdot^{\mathcal{I}_i^+} \mid i \in \omega\}$ is infinite as $\{\cdot^{\mathcal{I}_0^+}, \cdot^{\mathcal{I}_1^+}, \cdot^{\mathcal{I}_2^+}, \cdot^{\mathcal{I}_3^+}, \ldots\}$.

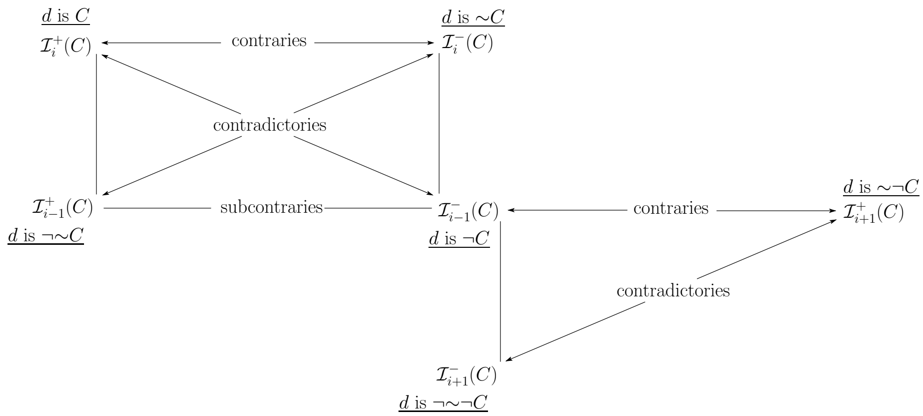


Fig. 3 Oppositions in $\mathcal{ALC}^n_{\sim}$-interpretations

follows:

$$
\begin{aligned}
(\neg C)^{\mathcal{I}_0^+} &= \Delta^{\mathcal{I}} \backslash C^{\mathcal{I}_0^+} & & \\
(\neg C)^{\mathcal{I}_i^+} &= C^{\mathcal{I}_{i-1}^-} (i > 0) & ({\sim}C)^{\mathcal{I}_i^+} &= C^{\mathcal{I}_i^-} \\
(C \sqcap D)^{\mathcal{I}_i^+} &= C^{\mathcal{I}_i^+} \cap D^{\mathcal{I}_i^+} & (C \sqcup D)^{\mathcal{I}_i^+} &= C^{\mathcal{I}_i^+} \cup D^{\mathcal{I}_i^+} \\
(\forall R.C)^{\mathcal{I}_i^+} &= \{d_1 \in \Delta^{\mathcal{I}} \mid \forall d_2[(d_1, d_2) \in R^{\mathcal{I}_0^+} \rightarrow d_2 \in C^{\mathcal{I}_i^+}]\} & & \\
(\exists R.C)^{\mathcal{I}_i^+} &= \{d_1 \in \Delta^{\mathcal{I}} \mid \exists d_2[(d_1, d_2) \in R^{\mathcal{I}_0^+} \wedge d_2 \in C^{\mathcal{I}_i^+}]\} & & \\
(\neg C)^{\mathcal{I}_i^-} &= C^{\mathcal{I}_{i+1}^+} & ({\sim}C)^{\mathcal{I}_i^-} &= C^{\mathcal{I}_i^+} \\
(C \sqcap D)^{\mathcal{I}_i^-} &= C^{\mathcal{I}_i^-} \cup D^{\mathcal{I}_i^-} & (C \sqcup D)^{\mathcal{I}_i^-} &= C^{\mathcal{I}_i^-} \cap D^{\mathcal{I}_i^-} \\
(\forall R.C)^{\mathcal{I}_i^-} &= \{d_1 \in \Delta^{\mathcal{I}} \mid \exists d_2[(d_1, d_2) \in R^{\mathcal{I}_0^+} \wedge d_2 \in C^{\mathcal{I}_i^-}]\} & & \\
(\exists R.C)^{\mathcal{I}_i^-} &= \{d_1 \in \Delta^{\mathcal{I}} \mid \forall d_2[(d_1, d_2) \in R^{\mathcal{I}_0^+} \rightarrow d_2 \in C^{\mathcal{I}_i^-}]\} & &
\end{aligned}
$$

An $\mathcal{ALC}^n_{\sim}$-interpretation $\mathcal{I}$ satisfies the contrary condition if for all concept names A, $A^{\mathcal{I}_0^+} \cup A^{\mathcal{I}_0^-} \neq \Delta^{\mathcal{I}}$, $A^{\mathcal{I}_{i+1}^+} \subsetneq A^{\mathcal{I}_i^+}$, and $A^{\mathcal{I}_{i+1}^-} \subsetneq A^{\mathcal{I}_i^-}$. In the two types of interpretations, conceptual oppositions are characterized as shown in Figure 2 and Figure 3. The $\mathcal{ALC}^2_{\sim}$-interpretation is defined as $({\sim}\neg C)^{\mathcal{I}^+} = (\neg C)^{\mathcal{I}^-} = C^{\mathcal{I}^+}$, and hence, $d \in ({\sim}\neg C)^{\mathcal{I}}$ if and only if $d \in C^{\mathcal{I}}$. This semantically causes loss in distinction between contraries (${\sim}\neg C$ and $\neg C$) and contradictories (C and $\neg C$). Instead, the $\mathcal{ALC}^n_{\sim}$-interpretation includes the definition $({\sim}\neg C)^{\mathcal{I}_i^+} = (\neg C)^{\mathcal{I}_i^-} = C^{\mathcal{I}_{i+1}^+}$ and $(\neg C)^{\mathcal{I}_i^+} = C^{\mathcal{I}_{i-1}^-}$ ($i > 0$), where infinite interpretation functions are required. That is, the $\mathcal{ALC}^n_{\sim}$-interpretation is improved to capture the oppositions – contraries, contradictories, and subcontraries in the philosophical study of negation.[6)]

Let $* \in \{2, n\}$. Each $\mathcal{ALC}^*_\sim$-concept is interpreted by $C^{\mathcal{I}^+}$ or $C^{\mathcal{I}^+_0}$ (denoted $C^{\mathcal{I}}$). Let C, D be $\mathcal{ALC}^*_\sim$-concepts. The TBox (terminological knowledge) is a finite set of general inclusion axioms of the form $C \sqsubseteq D$. We write a concept equation $C \equiv D$ as $C \sqsubseteq D$ and $D \sqsubseteq C$. An $\mathcal{ALC}^*_\sim$-concept C or a concept equation $C \equiv D$ is $\mathcal{ALC}^*_\sim$-satisfiable if there exists an $\mathcal{ALC}^*_\sim$-interpretation $\mathcal{I}$, called an $\mathcal{ALC}^*_\sim$-model of C (or $C \equiv D$), such that $C^{\mathcal{I}} \neq \emptyset$ (or $C^{\mathcal{I}} = D^{\mathcal{I}}$). Otherwise, it is $\mathcal{ALC}^*_\sim$-unsatisfiable. In particular, if an $\mathcal{ALC}^*_\sim$-concept C is $\mathcal{ALC}^*_\sim$-satisfiable and the $\mathcal{ALC}^*_\sim$-model satisfies the contrary condition, then it is $\mathcal{ALC}^*_\sim$-satisfiable under the contrary condition. Otherwise, it is $\mathcal{ALC}^*_\sim$-unsatisfiable under the contrary condition. A concept equation $C \equiv D$ is $\mathcal{ALC}^*_\sim$-valid if every $\mathcal{ALC}^*_\sim$-interpretation $\mathcal{I}$ is an $\mathcal{ALC}^*_\sim$-model of $C \equiv D$. We can derive the following fact from these interpretations:

Proposition 4.1
Let C be an $\mathcal{ALC}^2_\sim$-concept and D be an $\mathcal{ALC}^n_\sim$-concept. Then, the concept equation $C \equiv {\sim}\neg C$ is $\mathcal{ALC}^2_\sim$-valid, but the concept equation $D \equiv {\sim}\neg D$ is not $\mathcal{ALC}^n_\sim$-valid.

In addition, since $\mathcal{ALC}^n_\sim$-concepts do not contain the negation of roles, each role is interpreted only by the interpretation function $\cdot^{\mathcal{I}^+_0}$ (or $\cdot^{\mathcal{I}^+}$). Let us give an example of an $\mathcal{ALC}^n_\sim$-interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \{\cdot^{\mathcal{I}^+_i} \mid i \in \omega\}, \{\cdot^{\mathcal{I}^-_i} \mid i \in \omega\})$ such that $\Delta^{\mathcal{I}} = \{John, Mary, Tom\}$, $Happy^{\mathcal{I}^+_0} = \{John\}$, $Happy^{\mathcal{I}^-_0} = \{Mary, Tom\}$, $Happy^{\mathcal{I}^+_1} = \emptyset$, $Happy^{\mathcal{I}^-_1} = \{Tom\}$, $Happy^{\mathcal{I}^+_2} = \emptyset$, ... , $has\text{-}child^{\mathcal{I}^+_0} = \{(John, Tom)\}$ with $Happy^{\mathcal{I}^+_0} \cap Happy^{\mathcal{I}^-_0} = \emptyset$, $Happy^{\mathcal{I}^+_{i+1}} \subseteq Happy^{\mathcal{I}^+_i}$, and $Happy^{\mathcal{I}^-_{i+1}} \subseteq Happy^{\mathcal{I}^-_i}$. The interpretation functions $\cdot^{\mathcal{I}^+_i}$ and $\cdot^{\mathcal{I}^-_i}$ are expanded to the $\mathcal{ALC}^n_\sim$-concepts $\exists has\text{-}child.{\sim}Happy$, $\neg{\sim}\neg{\sim}Happy$ and $\neg{\sim}{\sim}\neg Happy$ as below.

$$\begin{aligned}(\exists has\text{-}child.{\sim}Happy)^{\mathcal{I}^+_0} &= \{d_1 \in \Delta^{\mathcal{I}} \mid \exists d_1[(d_1, d_2) \in has\text{-}child^{\mathcal{I}^+_0} \wedge d_2 \in Happy^{\mathcal{I}^-_0}]\}\\ &= \{John\}\end{aligned}$$

$$\begin{aligned}(\neg{\sim}\neg{\sim}Happy)^{\mathcal{I}^+_0} &= \Delta^{\mathcal{I}} \backslash ({\sim}\neg{\sim}Happy)^{\mathcal{I}^+_0}\\ &= \Delta^{\mathcal{I}} \backslash (\neg{\sim}Happy)^{\mathcal{I}^-_0}\\ &= \Delta^{\mathcal{I}} \backslash ({\sim}Happy)^{\mathcal{I}^+_1}\\ &= \Delta^{\mathcal{I}} \backslash Happy^{\mathcal{I}^-_1}\\ &= \{John, Mary\}\end{aligned} \qquad \begin{aligned}(\neg{\sim}{\sim}\neg Happy)^{\mathcal{I}^+_0} &= \Delta^{\mathcal{I}} \backslash ({\sim}{\sim}\neg Happy)^{\mathcal{I}^+_0}\\ &= \Delta^{\mathcal{I}} \backslash ({\sim}\neg Happy)^{\mathcal{I}^-_0}\\ &= \Delta^{\mathcal{I}} \backslash (\neg Happy)^{\mathcal{I}^+_0}\\ &= \{John\}\end{aligned}$$

Remark. Semantically, the three conditions $A^{\mathcal{I}_0^+} \cap A^{\mathcal{I}_0^-} = \emptyset$, $A^{\mathcal{I}_{i+1}^+} \subseteq A^{\mathcal{I}_i^+}$, and $A^{\mathcal{I}_{i+1}^-} \subseteq A^{\mathcal{I}_i^-}$ in the $\mathcal{ALC}^n_\sim$-interpretation $\mathcal{I}$ define the inconsistency of contraries between $\mathcal{ALC}^n_\sim$-concepts. Syntactically, the conditions lead to the inconsistent pairs $\langle A, \sim A\rangle$, $\langle \neg(\sim\neg)^i A, (\sim\neg)^{i+1} A\rangle$, and $\langle (\neg\sim)^{i+1} A, \sim(\neg\sim)^{i+1} A\rangle$ of $\mathcal{ALC}^n_\sim$-concepts. Each pair consists of a concept C and its strong negation $\sim C$ (i.e., $\langle C, \sim C\rangle$) where C is of the form A, $\neg(\sim\neg)^i A$, or $(\neg\sim)^{i+1} A$. For example, $\neg Red$ and $\sim\neg Red$ are inconsistent. In the next lemma, these conditions are generalized to any $\mathcal{ALC}^n_\sim$-concept.

Lemma 4.1
Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \{\cdot^{\mathcal{I}_i^+} \mid i \in \omega\}, \{\cdot^{\mathcal{I}_i^-} \mid i \in \omega\})$ be an $\mathcal{ALC}^n_\sim$-interpretation. For any $\mathcal{ALC}^n_\sim$-concept C, the following statements hold:

1. $C^{\mathcal{I}} \cap \sim C^{\mathcal{I}} = \emptyset$,
2. $(\sim\neg)^{i+1} C^{\mathcal{I}} \subseteq (\sim\neg)^i C^{\mathcal{I}}$,
3. $\sim(\neg\sim)^{i+1} C^{\mathcal{I}} \subseteq \sim(\neg\sim)^i C^{\mathcal{I}}$.

Proof. We show this lemma by induction on the structure of an $\mathcal{ALC}^n_\sim$-concept C.

Case 1: If $C = A$, then it is straightforward.

Case 2: $C = \neg C_1$. (i) Let $d \in (\neg C_1)^{\mathcal{I}_0^+}$. Then $d \notin C_1^{\mathcal{I}_0^+}$. By the induction hypothesis, we have $d \notin (\sim\neg C_1)^{\mathcal{I}_0^+}$ (by statement 2). (ii) Let $d \in (\sim\neg)^{i+1}\neg C_1^{\mathcal{I}_0^+}$. By Definition 4.2, $d \in \sim(\neg\sim)^i C_1^{\mathcal{I}_0^+}$. If $i = 0$, then $d \in \sim C_1^{\mathcal{I}_0^+}$. By the induction hypothesis, $d \notin C_1^{\mathcal{I}_0^+}$ (by statement 1). Hence, $d \in \neg C_1^{\mathcal{I}_0^+}$, that is, $d \in (\sim\neg)^i \neg C_1^{\mathcal{I}_0^+}$ $(i = 0)$. If $i > 0$, then $d \in \sim(\neg\sim)^{i-1} C_1^{\mathcal{I}_0^+}$ (by statement 3). So, we have $d \in (\sim\neg)^i \neg C_1^{\mathcal{I}_0^+}$. (iii) Let $d \notin \sim(\neg\sim)^i \neg C_1^{\mathcal{I}_0^+}$. By the induction hypothesis, $d \notin \sim(\neg\sim)^{i+1} \neg C_1^{\mathcal{I}_0^+}$ (by statement 2).

Case 3: $C = \sim C_1$. (i) Let $d \in (\sim C_1)^{\mathcal{I}_0^+}$. By the induction hypothesis, $d \notin C_1^{\mathcal{I}_0^+}$ (by statement 1). By Definition 4.2, it suffices for $d \notin (\sim\sim C_1)^{\mathcal{I}_0^+}$. (ii) Let $d \in (\sim\neg)^{i+1}\sim C_1^{\mathcal{I}_0^+} (= \sim(\neg\sim)^{i+1} C_1^{\mathcal{I}_0^+})$. By the induction hypothesis, $d \in \sim(\neg\sim)^i C_1^{\mathcal{I}_0^+}$ (by statement 3). That is, $d \in (\sim\neg)^i \sim C_1^{\mathcal{I}_0^+}$. (iii) Let $d \notin \sim(\neg\sim)^i \sim C_1^{\mathcal{I}_0^+}$. We can view it as $d \notin (\sim\neg)^i C_1^{\mathcal{I}_0^+}$. Thus, $d \notin (\sim\neg)^{i+1} C_1^{\mathcal{I}_0^+}$ by the induction hypothesis for (ii). Hence, we obtain $d \notin \sim(\neg\sim)^{i+1}\sim C_1^{\mathcal{I}_0^+}$.

Furthermore, if C is of the form $C_1 \sqcap C_2$, $C_1 \sqcup C_2$, $\forall R.C_1$, or $\exists R.C_1$, then the statements can be proved in a way similar to the one above. ∎

This lemma will be used to prove the correspondence between a tableau for an $\mathcal{ALC}^n_\sim$-concept and the satisfiability of the concept.

Any $\mathcal{ALC}^n_\sim$-concept is transformed into an equivalent one in a negation normal form (that is more complex than the negation normal form in $\mathcal{ALC}$) using the following equivalences, from left to right:

$$\begin{aligned}
(\neg)^k(\sim\neg)^i{\sim}{\sim}C &\equiv (\neg)^k(\sim\neg)^i C\\
(\sim)^k(\neg\sim)^i\neg\neg C &\equiv (\sim)^k(\neg\sim)^i C\\
(\neg)^k(\sim\neg)^i{\sim}(C \sqcap D) &\equiv (\neg)^k(\sim\neg)^i({\sim}C \sqcup {\sim}D)\\
(\neg)^k(\sim\neg)^i{\sim}(C \sqcup D) &\equiv (\neg)^k(\sim\neg)^i({\sim}C \sqcap {\sim}D)\\
(\neg)^k(\sim\neg)^i{\sim}(\forall R.C) &\equiv (\neg)^k(\sim\neg)^i(\exists R.{\sim}C)\\
(\neg)^k(\sim\neg)^i{\sim}(\exists R.C) &\equiv (\neg)^k(\sim\neg)^i(\forall R.{\sim}C)\\
(\sim)^k(\neg\sim)^i\neg(C \sqcap D) &\equiv (\sim)^k(\neg\sim)^i(\neg C \sqcup \neg D)\\
(\sim)^k(\neg\sim)^i\neg(C \sqcup D) &\equiv (\sim)^k(\neg\sim)^i(\neg C \sqcap \neg D)\\
(\sim)^k(\neg\sim)^i\neg(\forall R.C) &\equiv (\sim)^k(\neg\sim)^i(\exists R.\neg C)\\
(\sim)^k(\neg\sim)^i\neg(\exists R.C) &\equiv (\sim)^k(\neg\sim)^i(\forall R.\neg C)
\end{aligned}$$

where $k \in \{0, 1\}$ and $i \in \omega$. The form of the concepts obtained by this transformation is called a *constructive negation normal form*, where the four types of negation forms $(\sim\neg)^{i+1}$, $\neg(\sim\neg)^i$, $(\neg\sim)^{i+1}$, and ${\sim}(\neg\sim)^i$ occur only in front of a concept name. For example, $({\sim}\neg A_1 \sqcup \neg{\sim}{\sim}\neg A_2) \sqcap {\sim}(\neg\sim)^4 A_3$ is in the constructive negation normal form.

Proposition 4.2
Every concept equation $C \equiv D$ in the translation is $\mathcal{ALC}^n_\sim$-valid.

Next, we will discuss an important property of $\mathcal{ALC}^n_\sim$-interpretations that is derived from the contrary condition.

Lemma 4.2
Let $\mathcal{I} = (\Delta^\mathcal{I}, \{\cdot^{\mathcal{I}_i^+} \mid i \in \omega\}, \{\cdot^{\mathcal{I}_i^-} \mid i \in \omega\})$ be an $\mathcal{ALC}^n_\sim$-interpretation that satisfies the contrary condition. For any $\mathcal{ALC}^n_\sim$-concept C, the following statements hold:

1. $C^{\mathcal{I}} \cup \sim C^{\mathcal{I}} \neq \Delta^{\mathcal{I}}$,
2. $(\sim\neg)^{i+1} C^{\mathcal{I}} \subsetneq (\sim\neg)^{i} C^{\mathcal{I}}$,
3. $\sim(\neg\sim)^{i+1} C^{\mathcal{I}} \subsetneq \sim(\neg\sim)^{i} C^{\mathcal{I}}$.

Proof. We show this lemma by induction on the structure of an $\mathcal{ALC}^n_\sim$-concept C.

Case 1: If $C = A$, then it is straightforward.

Case 2: $C = \neg C_1$. (i) By the induction hypothesis, there exists d such that $d \in (\neg C_1)^{\mathcal{I}_0^+}$ and $d \notin (\sim\neg\neg C_1)^{\mathcal{I}_0^+}$ (by statement 2). Hence, $d \notin C_1^{\mathcal{I}_0^+}$ and $d \notin (\sim C_1)^{\mathcal{I}_0^+}$. (ii) ($i = 0$) By the induction hypothesis, there exists d such that $d \notin (\neg C_1)^{\mathcal{I}_0^+}$ and $d \notin (\sim\neg C_1)^{\mathcal{I}_0^+}$ (by statement 1). Then, $d \in C_1^{\mathcal{I}_0^+}$. ($i > 0$) By the induction hypothesis, $(\sim\neg)^{i+2} C_1^{\mathcal{I}} \subsetneq (\sim\neg)^{i+1} C_1^{\mathcal{I}}$ (by statement 3). (iii) ($i = 0$) By the induction hypothesis, there exists d such that $d \notin (\neg\sim C_1)^{\mathcal{I}_0^+}$ and $d \notin (\sim\neg\sim C_1)^{\mathcal{I}_0^+}$ (by statement 1). So, $d \in (\sim C_1)^{\mathcal{I}_0^+}$. ($i > 0$) By statement 2, $((\sim\neg)^{i+1} \neg C_1)^{\mathcal{I}} \subsetneq ((\sim\neg)^{i} \neg C_1)^{\mathcal{I}}$.

Case 3: $C = \sim C_1$. (i) By the induction hypothesis, there exists d such that $d \notin C_1^{\mathcal{I}_0^+}$ and $d \notin (\sim C_1)^{\mathcal{I}_0^+}$ (by statement 1). Hence, $d \notin (\sim\sim C_1)^{\mathcal{I}_0^+}$. (ii) By the induction hypothesis, $(\sim(\neg\sim)^{i+1} C_1)^{\mathcal{I}} \subsetneq (\sim(\neg\sim)^{i} C_1)^{\mathcal{I}}$ (by statement 3). Thus, $((\sim\neg)^{i+1} \sim C_1)^{\mathcal{I}} \subsetneq ((\sim\neg)^{i} \sim C_1)^{\mathcal{I}}$. (iii) By the induction hypothesis, $((\sim\neg)^{i+1} C_1)^{\mathcal{I}} \subsetneq ((\sim\neg)^{i} C_1)^{\mathcal{I}}$ (by statement 2). Thus, $(\sim(\neg\sim)^{i+1} \sim C_1)^{\mathcal{I}} \subsetneq (\sim(\neg\sim)^{i} \sim C_1)^{\mathcal{I}}$.

For the other cases: C is of the forms $C_1 \sqcap C_2$, $C_1 \sqcup C_2$, $\forall R.C_1$, and $\exists R.C_1$, the statements can be proved. ∎

This lemma guarantees that the proposed semantics characterizes the differences between contradictories and contraries in every interpretation. From the lemma it follows that all $\mathcal{ALC}^n_\sim$-interpretations need to have an infinite domain. This is an important model-theoretic property of the proposed logic because it is quite different from the standard description logic $\mathcal{ALC}$ (where every $\mathcal{ALC}$-concept is satisfiable in a finite model).

The following theorem states the property of contradictoriness and contrariness for $\mathcal{ALC}^n_\sim$-interpretations. Contradictoriness and contrariness are preserved in an $\mathcal{ALC}^n_\sim$-interpretation $\mathcal{I}$ of $\mathcal{ALC}^n_\sim$, if $\sim C^{\mathcal{I}} \subsetneq \neg C^{\mathcal{I}}$.

Theorem 4.2 (Contradictoriness and contrariness for $\mathcal{ALC}^n_\sim$)
If an $\mathcal{ALC}^n_\sim$-interpretation satisfies the contrary condition, then contradictoriness and contrariness are preserved in the $\mathcal{ALC}^n_\sim$-interpretation.

Proof. By statement 1 of Lemma 4.2 and by the $\mathcal{ALC}^n_\sim$-validity of $C \sqcup \neg C \equiv \top$, this can be proved. ∎

We would like to apply the replacement property[20)] to conceptual representation and strong negation in $\mathcal{ALC}^n_\sim$. Knowledge base designers rewrite concepts by their equivalent concepts in the context of (conceptual) knowledge representation (e.g., rebuilding ontologies in the Semantic Web). However, the replacement property provides a limitation such that concepts can only be replaced by strongly equivalent concepts when various combinations of the two types of negation are used. Let C, D be $\mathcal{ALC}^n_\sim$-concepts. C and D are equivalent if, for every $\mathcal{ALC}^n_\sim$-interpretation $\mathcal{I}$, $C^\mathcal{I} = D^\mathcal{I}$. C and D are strongly equivalent if, for every $\mathcal{ALC}^n_\sim$-interpretation $\mathcal{I}$ and for every $i \in \omega$, $C^{\mathcal{I}_i^+} = D^{\mathcal{I}_i^+}$ and $C^{\mathcal{I}_i^-} = D^{\mathcal{I}_i^-}$. Let E be $\mathcal{ALC}^n_\sim$-concept. Then, E_C indicates that E has a subconcept C. The concept $E_{D/C}$ is obtained from E_C by replacing some occurrence of C with D.

Theorem 4.3 (Replacement for $\mathcal{ALC}^n_\sim$)
Let C, D be $\mathcal{ALC}^n_\sim$-concepts. If C and D are strongly equivalent, then E_C and $E_{D/C}$ are also equivalent.

Proof. Let $\mathcal{I}$ be an $\mathcal{ALC}^n_\sim$-interpretation and let $i \in \omega$. We consider that E_C is of the forms $(\sim\neg)^i C^\mathcal{I}$, $\sim(\neg\sim)^i C^\mathcal{I}$, $(\neg\sim)^i C^\mathcal{I}$, $\neg(\sim\neg)^i C^\mathcal{I}$, $E'_C \sqcap E$, $E'_C \sqcup E$, $\forall R.E'_C$, and $\exists R.E'_C$.

Case 1: If $E_C = (\sim\neg)^i C$, then $(\sim\neg)^i C^\mathcal{I} = C^{\mathcal{I}_i^+} = D^{\mathcal{I}_i^+} = (\sim\neg)^i D^\mathcal{I}$.

Case 2: If $E_C = \sim(\neg\sim)^i C$, then $\sim(\neg\sim)^i C^\mathcal{I} = C^{\mathcal{I}_i^-} = D^{\mathcal{I}_i^-} = \sim(\neg\sim)^i D^\mathcal{I}$.

Case 3: If $E_C = (\neg\sim)^i C$, then $\neg\sim(\neg\sim)^{i-1} C^\mathcal{I} = \Delta^\mathcal{I} \backslash C^{\mathcal{I}_{i-1}^-} = \Delta^\mathcal{I} \backslash D^{\mathcal{I}_{i-1}^-} = \neg\sim(\neg\sim)^{i-1} D^\mathcal{I}$.

Case 4: If $E_C = \neg(\sim\neg)^i C$, then $\neg(\sim\neg)^i C^\mathcal{I} = \Delta^\mathcal{I} \backslash C^{\mathcal{I}_i^+} = \Delta^\mathcal{I} \backslash D^{\mathcal{I}_i^+} = \neg(\sim\neg)^i D^\mathcal{I}$.

Case 5: If $E_C = E'_C \sqcap E'$, then $(E'_C \sqcap E')^\mathcal{I} = (E'_C)^\mathcal{I} \cap (E')^\mathcal{I} = (E'_{D/C})^\mathcal{I} \cap (E')^\mathcal{I}$ (by the induction hypothesis) $= (E'_{D/C} \sqcap E')^\mathcal{I}$.

For the other forms $E'_C \sqcup E$, $\forall R.E'_C$, and $\exists R.E'_C$, the equivalence holds. ∎

It should be noted that the replacement property under strong equivalence is natural in the presence of strong negation. As a result of the semantics of $\mathcal{ALC}^n_\sim$, we obtain the property such that for any $\mathcal{ALC}^n_\sim$-concepts C, D, if C and D are equivalent, then C and D are strongly equivalent. Therefore, the following corollary is immediately derived from the replacement theorem.

Corollary 4.1
Let C, D be $\mathcal{ALC}^n_\sim$-concepts. If C and D are equivalent, then E_C and $E_{D/C}$ are also equivalent.

Proof. By assumption, for every $\mathcal{ALC}^n_\sim$-interpretation $\mathcal{I}$, $C^\mathcal{I} = D^\mathcal{I}$. So, by Definition 4.2, for every $\mathcal{ALC}^n_\sim$-interpretation $\mathcal{I}$ and for every $i \in \omega$, $C^{\mathcal{I}_i^+} = D^{\mathcal{I}_i^+}$, and $C^{\mathcal{I}_i^-} = D^{\mathcal{I}_i^-}$. By Theorem 4.3, the conlusion is obtained. ∎

This property enables a user to replace a subconcept (a concept included in a complex concept) with any equivalent concept (when rebuilding ontologies). In contrast, the description logic $\mathcal{ALC}^2_\sim$ using the conventional semantics of strong negation does not have this property. For example, the $\mathcal{ALC}^2_\sim$-concepts *Happy* and $\sim\neg$*Happy* are equivalent, but not strongly equivalent. This is because $\sim$*Happy* and $\sim\sim\neg$*Happy* are not equivalent. Hence, $\sim(\underline{\sim\neg Happy} \sqcap Person)$ cannot be replaced by $\sim(\underline{Happy} \sqcap Person)$.

4.2 Constructive description logic with Heyting negation and strong negation: $\mathcal{CALC}^2_\sim$

We define a concept language (called $\mathcal{CALC}^2_\sim$), that is an extension of the constructive description logic $\mathcal{CALC}^{N4}$,[12] by combining Heyting negation and strong negation. The concepts in the language (called $\mathcal{CALC}^2_\sim$-concepts) are constructed by concept names A; role names R; the connectives $\sqcap$, $\sqcup$, $-$ (Heyting negation), and $\sim$ (strong negation); and the quantifiers $\forall$, $\exists$. Every concept name $A \in \mathbf{C}$ is a $\mathcal{CALC}^2_\sim$-concept. If R is a role name and C, D are $\mathcal{CALC}^2_\sim$-concepts, then $-C$, $\sim C$, $C \sqcap D$, $C \sqcup D$, $\forall R.C$, and $\exists R.C$ are $\mathcal{CALC}^2_\sim$-concepts. We give an interpretation of $\mathcal{CALC}^2_\sim$-concepts (called a $\mathcal{CALC}^2_\sim$-interpretation) as follows:

Definition 4.3
A $\mathcal{CALC}^2_\sim$-interpretation $\mathcal{I}$ is a tuple $(W, \preceq, \Delta^\mathcal{I}, \{\cdot^{\mathcal{I}_t^+} \mid t \in W\}, \{\cdot^{\mathcal{I}_t^-} \mid t \in W\})$,

where W is a set of worlds, $\Delta^{\mathcal{I}} = \{\Delta^{\mathcal{I}_t} \mid t \in W\}$ is the family of non-empty sets and $\cdot^{\mathcal{I}_t^+}$ and $\cdot^{\mathcal{I}_t^-}$ are interpretation functions for each world $t \in W$ ($A^{\mathcal{I}_t^+}, A^{\mathcal{I}_t^-} \subseteq \Delta^{\mathcal{I}}$, $R^{\mathcal{I}_t^+} \subseteq \Delta^{\mathcal{I}_t} \times \Delta^{\mathcal{I}_t}$, and $a^{\mathcal{I}_t^+} \in \Delta^{\mathcal{I}_t}$) such that:

1. $\bot^{\mathcal{I}_t^+} = \emptyset$ and $\top^{\mathcal{I}_t^+} = \Delta^{\mathcal{I}_t}$,
2. $A^{\mathcal{I}_t^+} \cap A^{\mathcal{I}_t^-} = \emptyset$,
3. if $t, t' \in W$ and $t \preceq t'$, then $\Delta^{\mathcal{I}_t} \subseteq \Delta^{\mathcal{I}_{t'}}$, $A^{\mathcal{I}_t^+} \subseteq A^{\mathcal{I}_{t'}^+}$, $A^{\mathcal{I}_t^-} \subseteq A^{\mathcal{I}_{t'}^-}$, and $R^{\mathcal{I}_t^+} \subseteq R^{\mathcal{I}_{t'}^+}$.

For every world $t \in W$, the interpretation functions $\cdot^{\mathcal{I}_t^+}$ and $\cdot^{\mathcal{I}_t^-}$ are expanded to $\mathcal{CALC}^2_{\sim}$-concepts as follows:

$$(-C)^{\mathcal{I}_t^+} = \{d \mid d \in \Delta^{\mathcal{I}_{t'}} \backslash C^{\mathcal{I}_{t'}^+} \text{ s.t. } t \preceq t'\} \qquad (\sim C)^{\mathcal{I}_t^+} = C^{\mathcal{I}_t^-}$$

$$(C \sqcap D)^{\mathcal{I}_t^+} = C^{\mathcal{I}_t^+} \cap D^{\mathcal{I}_t^+} \qquad (C \sqcup D)^{\mathcal{I}_t^+} = C^{\mathcal{I}_t^+} \cup D^{\mathcal{I}_t^+}$$

$$(\forall R.C)^{\mathcal{I}_t^+} = \{d_1 \in \Delta^{\mathcal{I}_t} \mid \forall t'[t \preceq t' \to \forall d_2 \in \Delta^{\mathcal{I}_{t'}}[(d_1, d_2) \in R^{\mathcal{I}_{t'}^+} \to d_2 \in C^{\mathcal{I}_{t'}^+}]]\}$$

$$(\exists R.C)^{\mathcal{I}_t^+} = \{d_1 \in \Delta^{\mathcal{I}_t} \mid \exists d_2 \in \Delta^{\mathcal{I}_{t'}}[(d_1, d_2) \in R^{\mathcal{I}_t^+} \wedge d_2 \in C^{\mathcal{I}_t^+}]\}$$

$$(-C)^{\mathcal{I}_t^-} = C^{\mathcal{I}_t^+} \qquad (\sim C)^{\mathcal{I}_t^-} = C^{\mathcal{I}_t^+}$$

$$(C \sqcap D)^{\mathcal{I}_t^-} = C^{\mathcal{I}_t^-} \cup D^{\mathcal{I}_t^-} \qquad (C \sqcup D)^{\mathcal{I}_t^-} = C^{\mathcal{I}_t^-} \cap D^{\mathcal{I}_t^-}$$

$$(\forall R.C)^{\mathcal{I}_t^-} = \{d_1 \in \Delta^{\mathcal{I}_t} \mid \exists d_2 \in \Delta^{\mathcal{I}_{t'}}[(d_1, d_2) \in R^{\mathcal{I}_t^+} \wedge d_2 \in C^{\mathcal{I}_t^-}]\}$$

$$(\exists R.C)^{\mathcal{I}_t^-} = \{d_1 \in \Delta^{\mathcal{I}_t} \mid \forall t'[t \preceq t' \to \forall d_2 \in \Delta^{\mathcal{I}_{t'}}[(d_1, d_2) \in R^{\mathcal{I}_{t'}^+} \to d_2 \in C^{\mathcal{I}_{t'}^-}]]\}$$

An $\mathcal{ALC}^n_{\sim}$-interpretation $\mathcal{I}$ satisfies the contrary condition if $A^{\mathcal{I}^+} \cup A^{\mathcal{I}^-} \neq \Delta^{\mathcal{I}}$, where $A^{\mathcal{I}^+} = \bigcup_{t \in W} A^{\mathcal{I}_t^+}$ and $A^{\mathcal{I}^-} = \bigcup_{t \in W} A^{\mathcal{I}_t^-}$. The $\mathcal{CALC}^2_{\sim}$-interpretation $C^{\mathcal{I}}$ of each $\mathcal{CALC}^2_{\sim}$-concept is given by $\bigcup_{t \in W} C^{\mathcal{I}_t^+}$. A $\mathcal{CALC}^2_{\sim}$-concept C (or a concept equation $C \equiv D$) is $\mathcal{CALC}^2_{\sim}$-satisfiable if there exists an $\mathcal{CALC}^2_{\sim}$-interpretation $\mathcal{I}$, called a $\mathcal{CALC}^2_{\sim}$-model of C (or $C \equiv D$), such that $C^{\mathcal{I}} \neq \emptyset$ (or $C^{\mathcal{I}} = D^{\mathcal{I}}$); otherwise, it is $\mathcal{CALC}^2_{\sim}$-unsatisfiable. In particular, if a $\mathcal{CALC}^2_{\sim}$-concept C is $\mathcal{CALC}^2_{\sim}$-satisfiable and the $\mathcal{CALC}^2_{\sim}$-model satisfies the contrary condition, then it is $\mathcal{CALC}^2_{\sim}$-satisfiable under the contrary condition. Otherwise, it is $\mathcal{CALC}^2_{\sim}$-unsatisfiable under the contrary condition. A concept equation $C \equiv D$ is $\mathcal{CALC}^2_{\sim}$-valid if every $\mathcal{CALC}^2_{\sim}$-interpretation $\mathcal{I}$ is a $\mathcal{CALC}^2_{\sim}$-model of $C \equiv D$. Contradictoriness and contrariness are preserved in an $\mathcal{CALC}^2_{\sim}$-interpretation of $\mathcal{CALC}^2_{\sim}$ if $\sim C^{\mathcal{I}} \subsetneq -C^{\mathcal{I}}$.

Theorem 4.4 (Contradictoriness and contrariness for $\mathcal{CALC}^2_\sim$) Contradictoriness and contrariness are not preserved in some $\mathcal{CALC}^2_\sim$-interpretations that satisfy the contrary condition.

Proof. We construct a $\mathcal{CALC}^2_\sim$-interpretation $\mathcal{I} = (W, \preceq, \Delta^{\mathcal{I}}, \{\cdot^{\mathcal{I}_t^+} \mid t \in W\}, \{\cdot^{\mathcal{I}_t^-} \mid t \in W\})$ such that:

$$
\begin{array}{lll}
W = \{t_1, t_2, t_3\} & \preceq = \{(t_1, t_1), (t_1, t_2), (t_2, t_2)\} & \\
\Delta^{\mathcal{I}}_{t_1} = \{d_1, d_2\} & \Delta^{\mathcal{I}}_{t_2} = \{d_0, d_1, d_2\} & \Delta^{\mathcal{I}}_{t_3} = \{d_3\} \\
A^{\mathcal{I}^+_{t_1}} = A^{\mathcal{I}^+_{t_2}} = \{d_1\} & A^{\mathcal{I}^-_{t_1}} = A^{\mathcal{I}^-_{t_2}} = \{d_2\} & A^{\mathcal{I}^+_{t_3}} = A^{\mathcal{I}^-_{t_3}} = \emptyset.
\end{array}
$$

It satisfies the contrary condition $A^{\mathcal{I}^+} \cup A^{\mathcal{I}^-} = \{d_1, d_2\} \neq \{d_0, d_1, d_2, d_3\} = \Delta^{\mathcal{I}}$. By this interpretation, we have

$$
\begin{array}{ll}
(\sim A)^{\mathcal{I}} = \{d_2\} & (-A)^{\mathcal{I}} = \{d_0, d_2\} \\
(\sim - A)^{\mathcal{I}} = \{d_1\} & (- - A)^{\mathcal{I}} = \{d_1\}.
\end{array}
$$

However, since $(\sim - A)^{\mathcal{I}} = (- - A)^{\mathcal{I}}$, contradictoriness and contrariness are not preserved in the $\mathcal{CALC}^2_\sim$-interpretation. ∎

Table 1 shows the contradictoriness and contrariness for $\mathcal{ALC}^n_\sim$ and $\mathcal{CALC}^2_\sim$. The $\mathcal{CALC}^2_\sim$-concepts can be used to represent predicate denial and predicate term negation that capture conceptual models or describe a certain domain of interest; however, the contradictoriness and contrariness are not preserved in some $\mathcal{CALC}^2_\sim$-interpretations. For $\mathcal{ALC}^n_\sim$-concepts, the contradictoriness and contrariness are preserved in every $\mathcal{ALC}^n_\sim$-interpretation since strong negation is suitably added to the classical description logic $\mathcal{ALC}$ without the undesirable equivalent $C \equiv \sim\neg C$ in the semantics. In the next section, the tableau-based satisfiability algorithm for $\mathcal{ALC}$ is extended to $\mathcal{ALC}^n_\sim$. This extension is based on the contradictoriness and contrariness for the $\mathcal{ALC}^n_\sim$-interpretations.

Table 1 Contradictoriness and contrariness for $\mathcal{ALC}^2_\sim$, $\mathcal{ALC}^n_\sim$, and $\mathcal{CALC}^2_\sim$

DLs	Contradictoriness and contrariness
$\mathcal{ALC}^2_\sim$	not preserved for every interpretation
$\mathcal{ALC}^n_\sim$	**preserved for every interpretation**
$\mathcal{CALC}^2_\sim$	not preserved for some interpretations

Additionally, we show that the replacement property holds for strongly equivalent $\mathcal{CALC}^2_\sim$-concepts. Let C, D be $\mathcal{CALC}^2_\sim$-concepts. C and D are equivalent if, for every $\mathcal{CALC}^2_\sim$-interpretation $\mathcal{I}$, $C^{\mathcal{I}} = D^{\mathcal{I}}$. Let $C^{\mathcal{I}^+}$ denote $\bigcup_{t \in W} C^{\mathcal{I}_t^+}$ and $C^{\mathcal{I}^-}$ denote $\bigcup_{t \in W} C^{\mathcal{I}_t^-}$. C and D are strongly equivalent if, for every $\mathcal{CALC}^2_\sim$-interpretation $\mathcal{I}$, $C^{\mathcal{I}^+} = D^{\mathcal{I}^+}$ and $C^{\mathcal{I}^-} = D^{\mathcal{I}^-}$.

Theorem 4.5 (Replacement for $\mathcal{CALC}^2_\sim$)
Let C, D be $\mathcal{CALC}^2_\sim$-concepts. If C and D are strongly equivalent, then E_C and $E_{D/C}$ are (strongly) equivalent.

Proof. We show this theorem by induction on the structure of E_C.

Case 1: If $E_C = C$, then it is straightforward.

Case 2: If $E_C = \sim E'_C$, then $(\sim E'_C)^{\mathcal{I}^+} = \bigcup_{t \in W} (\sim E'_C)^{\mathcal{I}_t^+} = \bigcup_{t \in W} (E'_C)^{\mathcal{I}_t^-} = \bigcup_{t \in W} (E'_{D/C})^{\mathcal{I}_t^-}$ (by the induction hypothesis) $= \bigcup_{t \in W} (\sim E'_{D/C})^{\mathcal{I}_t^+} = (\sim E'_{D/C})^{\mathcal{I}^+}$, and $(\sim E'_C)^{\mathcal{I}^-} = \bigcup_{t \in W} (\sim E'_C)^{\mathcal{I}_t^-} = \bigcup_{t \in W} (E'_C)^{\mathcal{I}_t^+} = \bigcup_{t \in W} (E'_{D/C})^{\mathcal{I}_t^+}$ (by the induction hypothesis) $= \bigcup_{t \in W} (\sim E'_{D/C})^{\mathcal{I}_t^-} = (\sim E'_{D/C})^{\mathcal{I}^-}$.

Case 3: If $E_C = -E'_C$, then $(-E'_C)^{\mathcal{I}^+} = \bigcup_{t \in W} (-E'_C)^{\mathcal{I}_t^+} = \bigcup_{t \in W} \{d \mid d \in \Delta^{\mathcal{I}_{t'}} \backslash (E'_C)^{\mathcal{I}_{t'}^+} \text{ s.t. } t \preceq t'\} = \bigcup_{t \in W} \{d \mid d \in \Delta^{\mathcal{I}_{t'}} \backslash (E'_{D/C})^{\mathcal{I}_{t'}^+} \text{ s.t. } t \preceq t'\}$ (by the induction hypothesis) $= \bigcup_{t \in W} (-E'_{D/C})^{\mathcal{I}_t^+} = (-E'_{D/C})^{\mathcal{I}^+}$, and $(-E'_C)^{\mathcal{I}^-} = \bigcup_{t \in W} (-E'_C)^{\mathcal{I}_t^-} = \bigcup_{t \in W} (E'_C)^{\mathcal{I}_t^+} = \bigcup_{t \in W} (E'_{D/C})^{\mathcal{I}_t^+}$ (by the induction hypothesis) $= \bigcup_{t \in W} (-E'_{D/C})^{\mathcal{I}_t^-} = (-E'_{D/C})^{\mathcal{I}^-}$.

Case 4: If $E_C = E'_C \sqcap E'$, then $(E'_C \sqcap E')^{\mathcal{I}^+} = \bigcup_{t \in W} (E'_C \sqcap E')^{\mathcal{I}_t^+} = \bigcup_{t \in W} (E'_C)^{\mathcal{I}_t^+} \cap (E')^{\mathcal{I}_t^+} = \bigcup_{t \in W} (E'_{D/C})^{\mathcal{I}_t^+} \cap (E')^{\mathcal{I}_t^+}$ (by the induction hypothesis) $= \bigcup_{t \in W} (E'_{D/C} \sqcap E')^{\mathcal{I}_t^+} = (E'_{D/C} \sqcap E')^{\mathcal{I}^+}$, and $(E'_C \sqcap E')^{\mathcal{I}^-} = \bigcup_{t \in W} (E'_C \sqcap E')^{\mathcal{I}_t^-} = \bigcup_{t \in W} (E'_C)^{\mathcal{I}_t^-} \cup (E')^{\mathcal{I}_t^-} = \bigcup_{t \in W} (E'_{D/C})^{\mathcal{I}_t^-} \cup (E')^{\mathcal{I}_t^-}$ (by the induction hypothesis)

$$= \bigcup_{t \in W} (E'_{D/C} \sqcap E')^{\mathcal{I}_t^-} = (E'_{D/C} \sqcap E')^{\mathcal{I}^-}.$$

For the other forms $E'_C \sqcup E$, $\forall R.E'_C$, and $\exists R.E'_C$, the statement holds. ∎

§5 Tableau-based algorithm for $\mathcal{ALC}^n_\sim$

We denote $rol(C)$ as the set of roles occurring in an $\mathcal{ALC}^n_\sim$-concept C. For instance, $rol(\neg\forall R_1.\exists R_2.C_1 \sqcup \sim C_2) = \{R_1, R_2\}$. First we define a tableau for an $\mathcal{ALC}^n_\sim$-concept that is created by adding conditions for the forms $\sim C$ and $(\sim\neg)^i C$ to a tableau for an $\mathcal{ALC}$-concept.[7)]

Definition 5.1

Let D be an $\mathcal{ALC}^n_\sim$-concept in the constructive normal negation form. A tableau T for D is a tuple (S, L, E), where S is a set of individuals, $L\colon S \to 2^{sub(D)}$ is a mapping from each individual into a set of concepts in $sub(D)$, and $E\colon rol(D) \to 2^{S \times S}$ is a mapping from each role into a set of pairs of individuals. There exists some $s_0 \in S$ such that $D \in L(s_0)$, and for all $s, t \in S$, the following conditions hold:

1. if $C \in L(s)$, then $\sim C, \neg C \notin L(s)$,
2. if $C_1 \sqcap C_2 \in L(s)$, then $C_1 \in L(s)$ and $C_2 \in L(s)$,
3. if $C_1 \sqcup C_2 \in L(s)$, then $C_1 \in L(s)$ or $C_2 \in L(s)$,
4. if $\forall R.C \in L(s)$ and $(s,t) \in E(R)$, then $C \in L(t)$,
5. if $\exists R.C \in L(s)$, then there exists $t \in S$ such that $(s,t) \in E(R)$ and $C \in L(t)$,
6. for every $i \in \omega$, if $(\sim\neg)^{i+1} C \in L(s)$, then $(\sim\neg)^i C \in L(s)$.

In particular, it is called a C-tableau if the the following conditions hold:

7. for every $i \in \omega$ and for any $\mathcal{ALC}^n_\sim$-concept C, there exists $s \in S$ such that $(\sim\neg)^i C \in L(s)$ and $(\sim\neg)^{i+1} C \notin L(s)$,
8. for any $\mathcal{ALC}^n_\sim$-concept C, there exists $s \in S$ such that $C \notin L(s)$ and $\sim C \notin L(s)$.

Conditions 1 and 6 reflect the $\mathcal{ALC}^n_\sim$-interpretation of $\mathcal{ALC}^n_\sim$-concepts combining classical and strong negations. Condition 1 states that $C \in L(s)$ implies $\sim C \notin L(s)$ (in addition to $\neg C \notin L(s)$) to satisfy the semantic condition $A^{\mathcal{I}_0^+} \cap A^{\mathcal{I}_0^-} = \emptyset$. Moreover, Condition 6 is imposed for the semantic conditions $A^{\mathcal{I}^+_{i+1}} \subseteq A^{\mathcal{I}^+_i}$ and $A^{\mathcal{I}^-_{i+1}} \subseteq A^{\mathcal{I}^-_i}$. For example, by Condition 6, if $\sim\neg\sim$*Happy* $\in L(s)$, then $\sim$*Happy* $\in L(s)$. In the corresponding semantics, if $d \in (\sim\neg\sim Happy)^{\mathcal{I}^+_0}$, then $d \in (\sim Happy)^{\mathcal{I}^+_1}$. Hence, by the condition $A^{\mathcal{I}^+_{i+1}} \subseteq A^{\mathcal{I}^+_i}$, we get $d \in (\sim Happy)^{\mathcal{I}^+_0}$. Conditions 7 and 8 correspond to the contrary condition for the $\mathcal{ALC}^n_\sim$-interpretation, i.e. $A^{\mathcal{I}^+_0} \cup A^{\mathcal{I}^-_0} \neq \Delta^{\mathcal{I}}$, $A^{\mathcal{I}^+_{i+1}} \subsetneq A^{\mathcal{I}^+_i}$, and $A^{\mathcal{I}^-_{i+1}} \subsetneq A^{\mathcal{I}^-_i}$. The next lemma shows the correspondence between the existence of a tableau for an $\mathcal{ALC}^n_\sim$-concept and its satisfiability.

Lemma 5.1
Let D be an $\mathcal{ALC}^n_\sim$-concept. There exists a tableau for D if and only if it is $\mathcal{ALC}^n_\sim$-satisfiable. In particular, there exists a C-tableau for D if and only if it is $\mathcal{ALC}^n_\sim$-satisfiable under the contrary condition.

Proof. ($\Rightarrow$) Suppose we have a tableau $T = (S, L, E)$ for D. Then we can define an $\mathcal{ALC}^n_\sim$-model $\mathcal{I} = (\Delta^{\mathcal{I}}, \{\cdot^{\mathcal{I}^+_i} \mid i \in \omega\}, \{\cdot^{\mathcal{I}^-_i} \mid i \in \omega\})$ of D as follows:

$$\Delta^{\mathcal{I}} = S$$
$$A^{\mathcal{I}^+_i} = \{s \mid (\sim\neg)^i A \in L(s)\} \text{ for all } A \in sub(D)$$
$$A^{\mathcal{I}^-_i} = \{s \mid \sim(\neg\sim)^i A \in L(s)\} \text{ for all } A \in sub(D)$$
$$R^{\mathcal{I}^+_0} = E(R) \text{ for all } R \in rol(D).$$

For $\mathcal{I}$, we want to verify that for all the concepts C in $sub(D)$, $C \in L(s)$ implies $s \in C^{\mathcal{I}^+_0}$.

Case 1: If $C = A$, then by definition, $s \in C^{\mathcal{I}^+_0}$.

Case 2: If $C = (\sim\neg)^{i+1}A$ or $C = \sim(\neg\sim)^i A$, then by definition of the model and Definition 4.2, $s \in C^{\mathcal{I}^+_0}$.

Case 3: If $C = \neg(\sim\neg)^i A$, then by Condition 1 (in Definition 5.1), $(\sim\neg)^i A \notin L(s)$. By definition, we have $s \notin A^{\mathcal{I}_{(\sim\neg)^i}}$. So, $s \notin (\sim\neg)^i A^{\mathcal{I}^+_0}$ iff $s \in \Delta^{\mathcal{I}} \backslash (\sim\neg)^i A^{\mathcal{I}^+_0}$ iff $s \in \neg(\sim\neg)^i A^{\mathcal{I}^+_0}$.

Case 4: If $C = (\neg\sim)^{i+1}A$, then by Condition 1, $\sim(\neg\sim)^i A \notin L(s)$. So $s \notin A^{\mathcal{I}_{\sim(\neg\sim)^i}}$, and hence $s \in (\neg\sim)^{i+1}A^{\mathcal{I}^+_0}$. For the other cases of the forms $C_1 \sqcap C_2$, $C_1 \sqcup C_2$, $\forall R.C_1$, and $\exists R.C_1$, by Conditions 2-5, the claim holds. In addition, we will show that $\mathcal{I}$ is an $\mathcal{ALC}^n_\sim$-interpretation that satisfies the three conditions (i) $A^{\mathcal{I}^+_0} \cap A^{\mathcal{I}^-_0} = \emptyset$, (ii) $A^{\mathcal{I}^+_{i+1}} \subseteq A^{\mathcal{I}^+_i}$ and (iii) $A^{\mathcal{I}^-_{i+1}} \subseteq A^{\mathcal{I}^-_i}$. (i) Let $s \in A^{\mathcal{I}^+_0}$. Since $A \in L(s)$, by Condition 1, $\sim A \notin L(s)$. Hence $s \notin A^{\mathcal{I}^-_0}$. (ii) Let $s \in A^{\mathcal{I}^+_{i+1}}$. By definition, $(\sim\neg)^{i+1}A \in L(s)$. Then, by Condition 6,

$(\sim\neg)^i A \in L(s)$, and hence $s \notin \Delta^{\mathcal{I}} \backslash A^{\mathcal{I}_i^+}$. (iii) Let $s \in A^{\mathcal{I}_{i+1}^-}$. $s \notin \Delta^{\mathcal{I}} \backslash A^{\mathcal{I}_i^-}$ follows from Condition 6. Moreover, we show that if T is a C-tableau, then $\mathcal{I}$ satisfies the contrary condition. By Condition 8, there exists $s \in S$ such that $A \notin L(s)$ and $\sim A \notin L(s)$. By definition, $s \notin A^{\mathcal{I}_0^+}$ and $s \notin A^{\mathcal{I}_0^-}$. Hence, $A^{\mathcal{I}_0^+} \cup A^{\mathcal{I}_0^-} \neq \Delta^{\mathcal{I}}$. By Condition 7, there exists $s \in S$ such that $(\sim\neg)^i A \in L(s)$ and $(\sim\neg)^{i+1} A \notin L(s)$, and there exists $s' \in S$ such that $(\sim\neg)^i \sim A \in L(s')$ and $(\sim\neg)^{i+1} \sim A \notin L(s')$. By definition, $s \notin A^{\mathcal{I}_{i+1}^+}$ and $s \in A^{\mathcal{I}_i^+}$, and $s' \notin A^{\mathcal{I}_{i+1}^-}$ and $s' \in A^{\mathcal{I}_i^-}$.

($\Leftarrow$) Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \{\cdot^{\mathcal{I}_i^+} \mid i \in \omega\}, \{\cdot^{\mathcal{I}_i^-} \mid i \in \omega\})$ be an $\mathcal{ALC}^n_\sim$-model of D. Then we can construct a tableau $T = (S, L, E)$ for D as follows:

$$S = \Delta^{\mathcal{I}}$$

$$L(s) = \{C \in sub(D) \mid s \in C^{\mathcal{I}_0^+}\}$$

$$E(R) = R^{\mathcal{I}_0^+} \text{ for all } R \in rol(D).$$

Since D is $\mathcal{ALC}^n_\sim$-satisfiable, $D^{\mathcal{I}_0^+} \neq \theta$. So, by definition, $D \in L(s)$ for some $s \in S$ with $s \in D^{\mathcal{I}_0^+}$. By Lemma 4.1, it is inductively proved that T satisfies Conditions 1-6 in Definition 5.1. This shows T to be a tableau for D. Moreover, we show that if $\mathcal{I}$ satisfies the contrary condition, then T is a C-tableau. By Lemma 4.2, $C^{\mathcal{I}} \cup \sim C^{\mathcal{I}} \neq \Delta^{\mathcal{I}}$ and $(\sim\neg)^{i+1} C^{\mathcal{I}} \subsetneq (\sim\neg)^i C^{\mathcal{I}}$. Therefore, Conditions 7 and 8 in Definition 5.1 are verified. ∎

Lemma 5.1 indicates that given a tableau for an $\mathcal{ALC}^n_\sim$-concept D, we can define an $\mathcal{ALC}^n_\sim$-interpretation $\mathcal{I}$ satisfying it (i.e., an $\mathcal{ALC}^n_\sim$-model of D). The model is constructed in such a manner that for every constructive double negation $(\sim\neg)^i A$ (resp. $\sim(\neg\sim)^i A$) in $sub(D)$, $A^{\mathcal{I}_i^+}$ (resp. $A^{\mathcal{I}_i^-}$) is defined by the set of individuals $\{s \mid (\sim\neg)^i A \in L(s)\}$ (resp. $\{s \mid \sim(\neg\sim)^i A \in L(s)\}$).

To determine the satisfiability of $\mathcal{ALC}^n_\sim$-concepts, the tableau-based algorithm for $\mathcal{ALC}$ will be extended by introducing three new completion rules ($(\sim\neg)^i$-rule 1, $(\sim\neg)^i$-rule 2, and $\sim$-rule) and clash forms with respect to strong negation and constructive double negation. In Figure 4, the completion rules for $\mathcal{ALC}^n_\sim$-concepts are presented (as in Hollunder et al. and Schmidt-Schauss and Smolka[5, 16]). $(\sim\neg)^i$-rule 1 is applied to $\mathcal{ALC}^n_\sim$-concepts of the forms $(\sim\neg)^i A$ and $\sim(\neg\sim)^i A$. $(\sim\neg)^i$-rule 2 and $\sim$-rule introduce new variables if there exists no $z \in S$ such that $(\sim\neg)^{i+1} C \notin L(z)$ and $(\sim\neg)^i C \in L(z)$; or $\{A, \sim A\} \cap L(z) = \emptyset$.

$\sqcap$-rule: $L(x) = L(x) \cup \{C_1, C_2\}$
if $C_1 \sqcap C_2 \in L(x)$ and $\{C_1, C_2\} \not\subseteq L(x)$

$\sqcup$-rule: $L(x) = L(x) \cup \{C_1\}$ or $L(x) = L(x) \cup \{C_2\}$
if $C_1 \sqcup C_2 \in L(x)$ and $\{C_1, C_2\} \cap L(x) = \emptyset$

$\forall$-rule: $L(y) = L(y) \cup \{C\}$
if $\forall R.C \in L(x)$, $(x, y) \in E(R)$ and $C \notin L(y)$

$\exists$-rule: $S = S \cup \{y\}$ with $y \notin S$, $E(R) = E(R) \cup \{(x, y)\}$ and
$L(y) = \{C\}$
if $\exists R.C \in L(x)$ and $\{z \mid (x, z) \in E(R), C \in L(z)\} = \emptyset$

$(\sim\neg)^i$-rule 1: $L(x) = L(x) \cup \{(\sim\neg)^i C\}$
if $(\sim\neg)^{i+1} C \in L(x)$ and $(\sim\neg)^i C \notin L(x)$

$(\sim\neg)^i$-rule 2: $S = S \cup \{y\}$ with $y \notin S$ and $L(y) = \{(\sim\neg)^i C\}$
if $(\sim\neg)^i C \in L(x)$ and there exists no $z \in S$ such that
$(\sim\neg)^{i+1} C \notin L(z)$ and $(\sim\neg)^i C \in L(z)$

$\sim$-rule: $S = S \cup \{y\}$ with $y \notin S$ and $L(y) = \emptyset$
if $A \in L(x)$ or $\sim A \in L(x)$ and
there exists no $z \in S$ such that $\{A, \sim A\} \cap L(z) = \emptyset$.

Fig. 4 Completion rules for $\mathcal{ALC}^n_\sim$-concepts

Remark. The new algorithm has to recognize additional clash forms besides $\{A, \neg A\}$ and $\{\bot\}$. $L(x)$ contains a clash if it contains $\{C_1, \neg C_1\}$, $\{C_2, \sim C_2\}$, or $\{\bot\}$, where C_1 is of the form $(\sim\neg)^i A$ or $\sim(\neg\sim)^i A$ and C_2 is of the form $(\neg\sim)^i A$ or $\neg(\sim\neg)^i A$. For example, if $\{\neg\sim\neg A_1, \sim\neg\sim\neg A_1\} \subseteq L(x_1)$, then it contains a clash.

We present a tableau-based satisfiability algorithm for $\mathcal{ALC}^n_\sim$. Given an $\mathcal{ALC}^n_\sim$-concept D, the following procedure constructs a forest $ST = (S, E_{rol(D)} \cup E_\sim \cup E_{(\sim\neg)^i}, x_0)$ for D, where S is a set of individuals, each node $x \in S$ is labeled as $L(x)$, $E_{rol(D)} = \{(x,y) \in E(R) \mid R \in rol(D)\}$ (each edge $(x,y) \in E(R)$ is labeled as R), $(x,y) \in E_\sim \Leftrightarrow y$ is introduced for $A \in L(x)$ or $\sim A \in L(x)$ in $\sim$-rule, $(x,y) \in E_{(\sim\neg)^i} \Leftrightarrow y$ is introduced for $(\sim\neg)^i C \in L(x)$ in $(\sim\neg)^i$-rule 2, and x_0 is the root. First, set the initial forest $ST = (\{x_0\}, \emptyset, x_0)$, where $S = \{x_0\}$, $L(x_0) = \{D\}$, $E(R) = \emptyset$ for all $R \in rol(D)$, and $E_\sim = E_{(\sim\neg)^i} = \emptyset$. Then, apply completion rules in Figure 4 to ST until none of the rules are applicable. A forest ST is called complete if any completion rule is not applicable to it. If there is a clash-free complete forest ST, then return "satisfiable," and otherwise return "unsatisfiable."

Note that while every $\mathcal{ALC}^n_\sim$-interpretation $\mathcal{I}$ consists of infinite interpretation functions $\cdot^{\mathcal{I}_i^+}$ and $\cdot^{\mathcal{I}_i^-}$ for $i \in \omega$, we do not need to construct an infinite $\mathcal{ALC}^n_\sim$-model of an $\mathcal{ALC}^n_\sim$-concept. In other words, the satisfiability of each $\mathcal{ALC}^n_\sim$-concept can be decided by finite interpretation functions because the number of connectives occurring in it is finite. For example, the satisfiability of $(\sim\neg)^m A$ can be decided by the maximum $2m+1$ of interpretation functions $\cdot^{\mathcal{I}_0^+}, \ldots, \cdot^{\mathcal{I}_m^+}, \cdot^{\mathcal{I}_0^-}, \ldots, \cdot^{\mathcal{I}_{m-1}^-}$. In the proof of the completeness theorem, the finite interpretation is theoretically expanded to an $\mathcal{ALC}^n_\sim$-model by adding other infinite interpretation functions $\cdot^{\mathcal{I}_m^+}, \cdot^{\mathcal{I}_{m+1}^+}, \ldots, \cdot^{\mathcal{I}_m^-}, \cdot^{\mathcal{I}_{m+1}^-}, \ldots$. Thus, the finiteness of a completion forest ST for an $\mathcal{ALC}^n_\sim$-concept will be helpful when providing the termination of our satisfiability algorithm.

We show the correctness of the tableau-based satisfiability algorithm under the contrary condition (soundness, completeness, and termination)[*5] and the complexity of the satisfiability problem. In particular, we have to observe the behavior of the new completion rules in the algorithm. Unlike the other completion rules, each application of the new completion rules does not subdivide

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[*5] By deleting $(\sim\neg)^i$-rule 2 and $\sim$-rule, the tableau-based algorithm simply decides the satisfiability of $\mathcal{ALC}^n_\sim$-concepts, i.e., it does not consider the contrary condition. It can be shown that the algorithm is complete in the class of $\mathcal{ALC}^n_\sim$-interpretations.

a concept. However, since $(\sim\neg)^i$-rules 1 and 2 add only a subconcept to each node, $\sim$-rule creates an empty node, and the number of variables introduced in $(\sim\neg)^i$-rule 2 and $\sim$-rule is bounded by polynomial, the termination can be established.

Theorem 5.1 (Satisfiability under the contrary condition)
Let D be an $\mathcal{ALC}^n_\sim$-concept. The following statements hold:

1. The tableau-based algorithm terminates.
2. The tableau-based algorithm constructs a clash-free complete forest for an $\mathcal{ALC}^n_\sim$-concept D if and only if D is $\mathcal{ALC}^n_\sim$-satisfiable under the contrary condition.
3. Satisfiability of $\mathcal{ALC}^n_\sim$-concepts is PSPACE-complete.

Theorem 5.1 is proved by the following claims.

Claim 1
The tableau-based algorithm terminates.

Proof. Suppose that $|D| = m$ for an $\mathcal{ALC}^n_\sim$-concept D. For any node $x \in S$ in the forest $ST = (S, E_{rol(D)} \cup E_\sim \cup E_{(\sim\neg)^i}, x_0)$ for D, $L(x)$ must be a subset of $sub(D)$. Notice that when $(\sim\neg)^i$-rule is applied to a node $L(x)$ including $(\sim\neg)^{i+1}C$, the added concept $(\sim\neg)^iC$ is limited to a subconcept of D. Then $|L(x)| \leq m$ since $|sub(D)| \leq |D|$. Let $Num_\sim(L(x))$ denote the number of the form $\sim A$ or A occurring in a node $L(x)$, let $Num_{(\sim\neg)^i}(L(x))$ denote the number of the form $(\sim\neg)^iC$ occurring in a node $L(x)$, and let $Num_R(L(x))$ denote the number of the form $\exists R.C$ occurring in a node $L(x)$. Each node $L(x)$ can have child nodes $L(y)$ where each edge (x, y) is in $E_{rol(D)}$, $E_\sim$, or $E_{(\sim\neg)^i}$. The number of child nodes of each node is at most m since $Num_\sim(L(x)) + Num_{(\sim\neg)^i}(L(x)) + Num_R(L(x)) \leq m$. Therefore, we can apply the completion rules at most m times to each node x with $L(x)$. Moreover, we will find the depth of the forest ST. For $\exists R.C \in L(x)$, an application of $\exists$-rule creates a new node y from a leaf x where $L(y)$ contains only the subconcept C. For $\forall R.C' \in L(x)$ with $(x, y) \in E(R)$, an application of $\forall$-rule adds the subconcept C' to $L(y)$. Then, we have $dep(L(y)) < dep(L(x))$ since for any $C_i \in L(y)$, $\exists R.C_i$ or $\forall R.C_i$ must belong to $L(x)$. For $(\sim\neg)^iA \in L(x)$ or $\sim(\neg\sim)^iA \in L(x)$, $(\sim\neg)^i$-rule 2 can be applied at most i times. For $A \in L(x)$ or $\sim A \in L(x)$, $\sim$-rule can be applied once. Hence, the depth of ST is at most m^2, because of $dep(D) \leq m$. This leads to the termination. ∎

Claim 2
If the tableau-based algorithm constructs a clash-free complete forest for an $\mathcal{ALC}^n_\sim$-concept D, then D is $\mathcal{ALC}^n_\sim$-satisfiable under the contrary condition.

Proof. Let $ST = (S, E_{rol(D)}, x_0)$ with L and E be a clash-free complete forest for D, which is constructed by the tableau-based algorithm. Let us define $S' = S \cup S^+$ and $L' = L \cup L^+$ where

$$S^+ = \{v_{\sim A} \mid \text{there exists no } x \in S \text{ s.t. } A \in L(x) \text{ or } {\sim}A \in L(x)\} \cup$$
$$\{v_{(\sim\neg)^i C} \mid \text{there exists no } x \in S \text{ s.t. } (\sim\neg)^i C \notin L(x) \text{ or } (\sim\neg)^{i+1} C \in L(x)\}$$
$$L^+ = \{(v_{\sim A}, \emptyset) \mid v_{\sim A} \in S^+\} \cup$$
$$\{(v_{(\sim\neg)^i C}, \{(\sim\neg)^k C \mid 0 \leq k \leq i\}) \mid v_{(\sim\neg)^i C} \in S^+\}.$$

We will show the tuple (S', L', E) to be a tableau for D. By definition of the initial forest, $D \in L(x_0)$ for the root $x_0 \in S$. The tableau has to satisfy Conditions 1-6 in Definition 5.1.

Condition 1: Let $C \in L(x)$. Since ST is clash free, both ${\sim}C$ and $\neg C$ are not in $L(x)$. Let $C \in L^+(x)$. By the definition of L^+, ${\sim}C$ and $\neg C$ are not in $L(x)$.

Conditions 2-5: These are satisfied by definition of $\sqcap$-rule, $\sqcup$-rule, $\forall$-rule, and $\exists$-rule.

Condition 6: If $(\sim\neg)^{i+1} C \in L(x)$, then by $(\sim\neg)^i$-rule, $(\sim\neg)^i C \in L(x)$. If $(\sim\neg)^{i+1} C \in L^+(x)$, then by definition, $(\sim\neg)^i C \in L(x)$.

Moreover, we want to show it to be a C-tableau.

Condition 7: Suppose that there exists $x \in S$ with $(\sim\neg)^i C \in L(x)$. By $(\sim\neg)^i$-rule 2, there exists $y \in S$ such that $(\sim\neg)^i C \in L(y)$ and $(\sim\neg)^{i+1} C \notin L(y)$ since any completion rule does not add $(\sim\neg)^{i+1} C$ to $L(y)$. Suppose that there exists no $x \in S$ with $(\sim\neg)^i C \in L(x)$. By $(\sim\neg)^i$-rule 2, there exists no $x \in S$ with $(\sim\neg)^{i+1} C \in L(x)$. Hence, by definition, $v_{(\sim\neg)^i C} \in S^+$, and so $(\sim\neg)^i C \in L^+(v_{(\sim\neg)^i C})$ and $(\sim\neg)^{i+1} C \notin L^+(v_{(\sim\neg)^i C})$.

Condition 8: Let $C = A$. Suppose that there exists $x \in S$ with $A \in L(x)$ or ${\sim}A \in L(x)$. By $\sim$-rule, there exists $y \in S$ such that $L(y) = \emptyset$. So, $A, {\sim}A \notin L(y)$. Suppose that there exists no $x \in S$ with $A \in L(x)$ or ${\sim}A \in L(x)$. By definition, $v_{\sim A} \in S^+$, and $L^+(v_{\sim A}) = \emptyset$. Let $C = \neg(\sim\neg)^i A$. By $(\sim\neg)^i$-rule 2, there exists $x \in S$ with $(\sim\neg)^{i+1} A \notin L(x)$ and $(\sim\neg)^i A \in L(x)$. Since ST is clash free, $\neg(\sim\neg)^i A \notin L(x)$. Let $C = (\neg\sim)^i A$ $(i > 0)$. By $(\sim\neg)^i$-rule 2, there exists $x \in S$ with ${\sim}(\neg\sim)^i A \notin L(x)$ and ${\sim}(\neg\sim)^{i-1} A \in L(x)$. Since ST is clash free, $(\neg\sim)^i A \notin L(x)$. Hence, $C, \sim C \notin L(x)$. Therefore, by Lemma 5.1, D has an $\mathcal{ALC}^n_\sim$-model that satisfies the contrary condition. ∎

Claim 3
If an $\mathcal{ALC}^n_\sim$-concept D is $\mathcal{ALC}^n_\sim$-satisfiable under the contrary condition, then the tableau-based algorithm constructs a clash-free complete forest for D.

Proof. Let $\mathcal{I} = (\Delta^\mathcal{I}, \{\cdot^{\mathcal{I}_i^+} \mid i \in \omega\}, \{\cdot^{\mathcal{I}_i^-} \mid i \in \omega\})$ be an $\mathcal{ALC}^n_\sim$-model of D that satisfies the contrary condition. By Lemma 5.1, there exists a C-tableau $T = (S, L, E)$ for D where $D \in L(s_0)$ for some $s_0 \in S$. The tableau-based algorithm constructs a forest $ST = (S', E'_{rol(D)}, x_0)$ with L' and E'. In the forest construction, we define a mapping $\pi\colon S' \to S$ such that (i) $\pi(x_0) = s_0$, (ii) if $\pi(x) = s$, $\exists R.C \in L'(x)$, and y is introduced in $\exists$-rule, then $\pi(y) = t$ for some $t \in S$ with $C \in L(t)$ and $(s, t) \in E(R)$, (iii) if $\pi(x) = s$, $(\sim\neg)^i C \in L'(x)$ and y is introduced in $(\sim\neg)^i$-rule 2, then $\pi(y) = t$ for some $t \in S$ with $(\sim\neg)^i C \in L(t)$ and $(\sim\neg)^{i+1} C \notin L(t)$, and (iv) if $\pi(x) = s$, $\{A, \sim A\} \cap L'(x) \neq \emptyset$ and y is introduced in $\sim$-rule, then $\pi(y) = t$ for some $t \in S$ with $A, \sim A \notin L(t)$. Now we verify $L'(x) \subseteq L(\pi(x))$ for all $x \in S'$ in ST. For the initial forest, $L'(x_0) = \{D\} \subseteq L(\pi(x_0)) = L(s_0)$. Let $(\sim\neg)^i$-rule be applied to $(\sim\neg)^{i+1} C \in L'(x)$. Then, $L'(x) = L'(x) \cup \{(\sim\neg)^i C\}$. By the induction hypothesis, $(\sim\neg)^{i+1} C \in L(\pi(x))$. By Condition 6, $L(\pi(x))$ contains $(\sim\neg)^k C$ for $0 \leq k \leq i$. Moreover, by Condition 1 and Claim 1, it follows that ST is clash free and complete. ∎

Claim 4
Satisfiability of $\mathcal{ALC}^n_\sim$-concepts is in PSPACE.

Proof. Similar to the proof in [3)], this can be shown. By checking a depth-first construction of a forest ST (the depth of which is found in the proof of Claim 1), we reuse space that is bounded by polynomial. It should be noted that the contrary condition indicates that infinite individuals must exist that define $A^{\mathcal{I}_0^+} \cup A^{\mathcal{I}_0^-} \neq \Delta^\mathcal{I}$, $A^{\mathcal{I}_{i+1}^+} \subsetneq A^{\mathcal{I}_i^+}$, and $A^{\mathcal{I}_{i+1}^-} \subsetneq A^{\mathcal{I}_i^-}$, but $(\sim\neg)^i$-rule 2 and $\sim$-rule introduce new variables whose number is bounded by polynomial for each node in a forest ST. The hardness of satisfiability is derived from $\mathcal{ALC}$.[16)] ∎

§6 Conclusion

We have presented an extended description logic $\mathcal{ALC}^n_\sim$ that incorporates classical negation and strong negation for representing contraries, contradictories, and subcontraries between concepts. Technically, the semantics of strong negation is adapted to the oppositions in the philosophical study of negation.[9)] The important specification of our description logic is that strong negation is added to the classical description logic $\mathcal{ALC}$ and the property of contradictoriness and contrariness holds for every interpretation. We have demonstrated that the two negations are adequately characterized by $\mathcal{ALC}^n_\sim$-interpretations, comparing it with other description logics $\mathcal{ALC}^2_\sim$ and $\mathcal{CALC}^2_\sim$ (which we have originally defined by introducing strong negation). We have developed a tableau-based satisfiability algorithm for $\mathcal{ALC}^n_\sim$ that is extended to add three new completion rules to the tableau-based algorithm for $\mathcal{ALC}$.[16)] It finds an $\mathcal{ALC}^n_\sim$-model satisfying the contrary condition, in which a constructive normal negation form and various clash forms are defined to treat complex negative concepts. Consequently, the description logic provides a decidable fragment (precisely, PSPACE-complete) of classical first-order logic with classical negation and strong negation (but not constructive description logic with Heyting negation and strong negation).

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